Wenxiong Chen and Congming Li

Methods on Nonlinear Elliptic Equations

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Preface

In this book we intend to present basic materials as well as real research examples to young researchers in the field of non-linear analysis for partial differential equations (PDEs), in particular, for semi-linear elliptic PDEs. We hope it will become a good reading material for graduate students and a handy textbook for professors to use in a topic course on non-linear analysis.

We will introduce a series of well-known typical methods in non-linear analysis. We will first illustrate them by using simple examples, then we will lead readers to the research front and explain how these methods can be applied to solve practical problems through careful analysis of a series of recent research articles, mostly the authors' recent papers.

From our experience, roughly speaking, in applying these commonly used methods, there are usually two aspects:

i) general scheme (more or less universal) and,

ii) key points for each individual problem.

We will use our own research articles to show readers what the general schemes are and what the key points are; and with the understandings of these examples, we hope the readers will be able to apply these general schemes to solve their own research problems with the discovery of their own key points.

In Chapter 1, we introduce basic knowledge on Sobolev spaces and some commonly used inequalities. These are the major spaces in which we will seek weak solutions of PDEs.

Chapter 2 shows how to find weak solutions for some typical linear and semi-linear PDEs by using functional analysis methods, mainly, the calculus of variations and critical point theories including the well-known *Mountain Pass Lemma*.

In Chapter 3, we establish $W^{2,p}$ a priori estimates and regularity. We prove that, in most cases, weak solutions are actually differentiable, and hence are classical ones. We will also present a *Regularity Lifting Theorem*. It is a simple method to boost the regularity of solutions. It has been used extensively in various forms in the authors' previous works. The essence of the approach is well-known in the analysis community. However, the version we present here contains some new developments. It is much more general and is very easy to use. We believe that our method will provide convenient ways, for both experts and non-experts in the field, in obtaining regularities. We will use examples to show how this theorem can be applied to PDEs and to integral equations.

Chapter 4 is a preparation for Chapter 5 and 6. We introduce Riemannian manifolds, curvatures, covariant derivatives, and Sobolev embedding on manifolds.

Chapter 5 deals with semi-linear elliptic equations arising from prescribing Gaussian curvature, on both positively and negatively curved manifolds. We show the existence of weak solutions in critical cases via variational approaches. We also introduce the method of lower and upper solutions.

Chapter 6 focus on solving a problem from prescribing scalar curvature on S^n for $n \geq 3$. It is in the critical case where the corresponding variational functional is not compact at any level sets. To recover the compactness, we construct a max-mini variational scheme. The outline is clearly presented, however, the detailed proofs are rather complex. The beginners may skip these proofs.

Chapter 7 is devoted to the study of various maximum principles, in particular, the ones based on comparisons. Besides classical ones, we also introduce a version of maximum principle at infinity and a maximum principle for integral equations which basically depends on the absolute continuity of a Lebesgue integral. It is a preparation for the method of moving planes.

In Chapter 8, we introduce the method of moving planes and its variant–the method of moving spheres–and apply them to obtain the symmetry, monotonicity, a priori estimates, and even non-existence of solutions. We also introduce an integral form of the method of moving planes which is quite different than the one for PDEs. Instead of using local properties of a PDE, we exploit global properties of solutions for integral equations. Many research examples are illustrated, including the well-known one by Gidas, Ni, and Nirenberg, as well as a few from the authors' recent papers.

Chapters 7 and 8 function as a self-contained group. The readers who are only interested in the maximum principles and the method of moving planes, can skip the first six chapters, and start directly from Chapter 7.

Yeshiva University *Wenxiong Chen*
University of Colorado *Congming Li*

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Introduction to Sobolev Spaces

In real life, people use numbers to quantify the surrounding objects. In mathematics, the absolute value $|a - b|$ is used to measure the difference between two numbers a and b. Functions are used to describe physical states. For example, temperature is a function of time and place. Very often, we use a sequence of approximate solutions to approach a real one; and how close these solutions are to the real one depends on how we measure them, that is, which metric we are choosing. Hence, it is imperative that we need not only develop suitable metrics to measure different states (functions), but we must also study relationships among different metrics. For these purposes, the Sobolev Spaces were introduced. They have many applications in various branches of mathematics, in particular, in the theory of partial differential equations.

The role of Sobolev Spaces in the analysis of PDEs is somewhat similar to the role of Euclidean spaces in the study of geometry. The fundamental research on the relations among various Sobolev Spaces (Sobolev norms) was first carried out by G. Hardy and J. Littlewood in the 1910s and then by S. Sobolev in the 1930s. More recently, many well known mathematicians, such as H. Brezis, L. Caffarelli, A. Chang, E. Lieb, L. Nirenberg, J. Serrin, and

E. Stein, have worked in this area. The main objectives are to determine if and how the norms dominate each other, what the sharp estimates are, which functions achieve these sharp estimates, and which functions are 'critically' related to these sharp estimates.

To find the existence of weak solutions for partial differential equations, especially for nonlinear partial differential equations, the method of functional analysis, in particular, the calculus of variations, has seen more and more applications.

To roughly illustrate this kind of application, let's start off with a simple example. Let Ω be a bounded domain in R^n and consider the Dirichlet problem associated with the Laplace equation:

$$\begin{cases} -\triangle u = f(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

To prove the existence of solutions, one may view $-\triangle$ as an operator acting on a proper linear space and then apply some known principles of functional analysis, for instance, the 'fixed point theory' or 'the degree theory,' to derive the existence. One may also consider the corresponding variational functional

$$J(u) = \frac{1}{2}\int_{\Omega} |\bigtriangledown u|^2 \, d\,x - \int_{\Omega} f(x)\, u \, d\,x \tag{1.2}$$

in a proper linear space, and seek critical points of the functional in that space. This kind of variational approach is particularly powerful in dealing with nonlinear equations. For example, in equation (1.1), instead of $f(x)$, we consider $f(x,u)$. Then it becomes a semi-linear equation. Correspondingly, we have the functional

$$J(u) = \frac{1}{2}\int_{\Omega} |\bigtriangledown u|^2 \, d\,x - \int_{\Omega} F(x,u) \, d\,x, \tag{1.3}$$

where

$$F(x,u) = \int_0^u f(x,s)\, d\,x$$

is an anti-derivative of $f(x,\cdot)$. From the definition of the functional in either (1.2) or (1.3), one can see that the function u in the space need not be second order differentiable as required by the classical solutions of (1.1). Hence the critical points of the functional are solutions of the problem only in the 'weak' sense. However, by an appropriate regularity argument, one may recover the differentiability of the solutions, so that they can still satisfy equation (1.1) in the classical sense.

In general, given a PDE problem, our intention is to view it as an operator A acting on some proper linear spaces X and Y of functions and to symbolically write the equation as

$$Au = f \tag{1.4}$$

Then we can apply the general and elegant principles of linear or nonlinear functional analysis to study the solvability of various equations involving A. The result can then be applied to a broad class of partial differential equations. We may also associate this operator with a functional $J(\cdot)$, whose critical points are the solutions of the equation (1.4). In this process, the key is to find an appropriate operator 'A' and appropriate spaces 'X' and 'Y'. As we will see later, the Sobolev spaces are designed precisely for this purpose and will work out properly.

In solving a partial differential equation, in many cases, it is natural to first find a sequence of approximate solutions, and then go on to investigate the convergence of the sequence. The limit function of a convergent sequence of approximate solutions would be the desired exact solution of the equation. As we will see in the next few chapters, there are two basic stages in showing convergence:

i) in a reflexive Banach space, every bounded sequence has a weakly convergent subsequence, and then

ii) by the compact embedding from a "stronger" Sobolev space into a "weaker" one, the weak convergent sequence in the "stronger" space becomes a strong convergent sequence in the "weaker" space.

Before going onto the details of this chapter, the readers may have a glance at the introduction of the next chapter to gain more motivations for studying the Sobolev spaces.

In Section 1.1, we will introduce the *distributions*, mainly the notion of the *weak derivatives*, which are the elements of the Sobolev spaces.

We then define *Sobolev spaces* in Section 1.2.

To derive many useful properties in Sobolev spaces, it is not so convenient to work directly on weak derivatives. Hence in Section 1.3, we show that, these weak derivatives can be approximated by smooth functions. Then in the next three sections, we can just work on smooth functions to establish a series of important inequalities.

1.1 Distributions

As we have seen in the introduction, for the functional $J(u)$ in (1.2) or (1.3), what really involved were only the first derivatives of u instead of the second derivatives as required classically for a second order equation; and these first derivatives need not be continuous or even be defined everywhere. Therefore, via functional analysis approach, one can substantially weaken the notion of partial derivatives. The advantage is that it allows one to divide the task of finding "suitable" smooth solutions for a PDE into two major steps:

Step 1. Existence of Weak Solutions. One seeks solutions which are less differentiable but are easier to obtain. It is very common that people use "energy" minimization or conservation, sometimes use finite dimensional approximation, to show the existence of such weak solutions.

Step 2. Regularity Lifting. One uses various analysis tools to boost the differentiability of the known weak solutions and try to show that they are actually classical ones.

Both the existence of weak solutions and the regularity lifting have become two major branches of today's PDE analysis. Various functions spaces and the related embedding theories are basic tools in both analysis, among which Sobolev spaces are the most frequently used ones.

In this section, we introduce the notion of 'weak derivatives,' which will be the elements of the Sobolev spaces.

Let R^n be the n-dimensional Euclidean space and Ω be an open connected subset in R^n. Let $D(\Omega) = C_0^\infty(\Omega)$ be the linear space of infinitely differentiable functions with compact support in Ω. This is called the space of test functions on Ω.

Example 1.1.1 *Assume*

$$B_R(x^o) := \{x \in R^n \mid |x - x^o| < R\} \subset \Omega,$$

then for any $r < R$, the following function

$$f(x) = \begin{cases} \exp\{\frac{1}{|x-x^o|^2 - r^2}\} & \text{for } |x - x^o| < r \\ 0 & \text{elsewhere} \end{cases}$$

is in $C_0^\infty(\Omega)$.

Example 1.1.2 *Assume $\rho \in C_0^\infty(R^n)$, $u \in L^p(\Omega)$, and $supp\, u \subset K \subset\subset \Omega$. Let*

$$u_\epsilon(x) := \rho_\epsilon * u := \int_{R^n} \frac{1}{\epsilon^n} \rho(\frac{x-y}{\epsilon}) u(y) dy.$$

Then $u_\epsilon \in C_0^\infty(\Omega)$ for ϵ sufficiently small.

Let $L^p_{\text{loc}}(\Omega)$ be the space of p^{th}-power locally summable functions for $1 \leq p \leq \infty$. Such functions are Lebesque measurable functions f defined on Ω and with the property that

$$\|f\|_{L^p(K)} := \left(\int_K |f(x)|^p dx \right)^{1/p} < \infty$$

for every compact subset K in Ω.

Assume that u is a C^1 function in Ω and $\phi \in D(\Omega)$. Through integration by parts, we have, for $i = 1, 2, \cdots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_i} \phi(x) dx = - \int_\Omega u(x) \frac{\partial \phi}{\partial x_i} dx. \tag{1.5}$$

Now if u is not in $C^1(\Omega)$, then $\dfrac{\partial u}{\partial x_i}$ does not exist. However, the integral on the right hand side of (1.5) still makes sense if u is a locally L^1 summable function. For this reason, we define the first derivative $\dfrac{\partial u}{\partial x_i}$ weakly as the function $v(x)$ that satisfies

$$\int_\Omega v(x)\phi(x)dx = -\int_\Omega u\frac{\partial \phi}{\partial x_i}dx$$

for all functions $\phi \in D(\Omega)$.

The same idea works for higher partial derivatives. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index of order

$$k := |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

For $u \in C^k(\Omega)$, the regular αth partial derivative of u is

$$D^\alpha u = \frac{\partial^{\alpha_1}\partial^{\alpha_2}\cdots\partial^{\alpha_n}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

Given any test function $\phi \in D(\Omega)$, through a straight forward integration by parts k times, we arrive at

$$\int_\Omega D^\alpha u\, \phi(x)\, d\,x = (-1)^{|\alpha|}\int_\Omega u\, D^\alpha \phi\, d\,x. \tag{1.6}$$

There is no boundary term because ϕ vanishes near the boundary.

Now if u is not k times differentiable, the left hand side of (1.6) makes no sense. However the right hand side is valid for functions u with much weaker differentiability, i.e. u only need to be locally L^1 summable. Thus it is natural to choose those functions v that satisfy (1.6) as the weak representatives of $D^\alpha u$.

Definition 1.1.1 *For $u, v \in L^1_{loc}(\Omega)$, we say that v is the αth weak derivative of u, written*

$$v = D^\alpha u$$

provided

$$\int_\Omega v(x)\, \phi(x)\, d\,x = (-1)^{|\alpha|}\int_\Omega u\, D^\alpha \phi\, d\,x$$

for all test functions $\phi \in D(\Omega)$.

Example 1.1.3 *For $n = 1$ and $\Omega = (-\pi, 1)$, let*

$$u(x) = \begin{cases} \cos x & \text{if } -\pi < x \leq 0 \\ 1 - x & \text{if } 0 < x < 1. \end{cases}$$

Then its weak derivative $u'(x)$ can be represented by

$$v(x) = \begin{cases} -\sin x & \text{if } -\pi < x \leq 0 \\ -1 & \text{if } 0 < x < 1. \end{cases}$$

To see this, we verify, for any $\phi \in D(\Omega)$, *that*

$$\int_{-\pi}^{1} u(x)\phi'(x)dx = -\int_{-\pi}^{1} v(x)\phi(x)dx. \tag{1.7}$$

In fact, through integration by parts, we have

$$\begin{aligned}\int_{-\pi}^{1} u(x)\phi'(x)dx &= \int_{-\pi}^{0} \cos x\,\phi'(x)dx + \int_{0}^{1}(1-x)\phi'(x)dx \\ &= \int_{-\pi}^{0} \sin x\,\phi(x)dx + \phi(0) - \phi(0) + \int_{0}^{1}\phi(x)dx \\ &= -\int_{-\pi}^{1} v(x)\phi(x)dx.\end{aligned}$$

In this example, one can see that, in the classical sense, the function u is not differentiable at $x = 0$. Since the weak derivative is defined by integrals, one may alter the values of the weak derivative $v(x)$ in a set of measure zero, and (1.7) still holds. However, it should be unique up to a set of measure zero.

Lemma 1.1.1 *(Uniqueness of Weak Derivatives). If* v *and* w *are the weak* α*th partial derivatives of* u, $D^\alpha u$, *then* $v(x) = w(x)$ *almost everywhere.*

Proof. By the definition of the weak derivatives, we have

$$\int_\Omega v(x)\phi(x)dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\phi dx = \int_\Omega w(x)\phi(x)dx$$

for any $\phi \in D(\Omega)$. It follows that

$$\int_\Omega (v(x) - w(x))\phi(x)dx = 0 \quad \forall \phi \in D(\Omega).$$

Therefore, we must have

$$v(x) = w(x) \quad \text{almost everywhere .}$$

□

From the definition, we can view a weak derivative as a linear functional acting on the space of test functions $D(\Omega)$, and we call it a distribution. More generally, we have

Definition 1.1.2 *A distribution is a continuous linear functional on* $D(\Omega)$. *The linear space of distributions or the generalized functions on* Ω, *denoted by* $D'(\Omega)$, *is the collection of all continuous linear functionals on* $D(\Omega)$.

Here, the continuity of a functional T on $D(\Omega)$ means that, for any sequence $\{\phi_k\} \subset D(\Omega)$ with $\phi_k \to \phi$ in $D(\Omega)$, we have

$$T(\phi_k) \to T(\phi), \quad \text{as } k \to \infty;$$

and we say that $\phi_k \to \phi$ in $D(\Omega)$ if

a) there exists $K \subset\subset \Omega$ such that $\text{supp}\phi_k$, $\text{supp}\phi \subset K$, and

b) for any α, $D^\alpha \phi_k \to D^\alpha \phi$ uniformly as $k \to \infty$.

The most important and most commonly used distributions are locally summable functions. In fact, for any $f \in L^p_{\text{loc}}(\Omega)$ with $1 \leq p \leq \infty$, consider

$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx.$$

It is easy to verify that $T_f(\cdot)$ is a continuous linear functional on $D(\Omega)$, and hence it is a distribution.

For any distribution μ, if there is an $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\mu(\phi) = T_f(\phi), \ \ \forall \phi \in D(\Omega),$$

then we say that μ is (or can be realized as) a locally summable function and identify it as f.

An interesting example of a distribution that is not a locally summable function is the well-known Dirac delta function. Let x^o be a point in Ω. For any $\phi \in D(\Omega)$, the delta function at x^o can be defined as

$$\delta_{x^o}(\phi) = \phi(x^o).$$

Hence it is a distribution. However, one can show that such a delta function is not locally summable. It is not a function at all. This kind of "function" has been used widely and so successfully by physicists and engineers, who often simply view δ_{x^o} as

$$\delta_{x^o}(x) = \begin{cases} 0, & \text{for } x \neq x^o \\ \infty, & \text{for } x = x^o. \end{cases}$$

Surprisingly, such a delta function is the derivative of some function in the following distributional sense. To explain this, let $\Omega = (-1, 1)$, and let

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \geq 0. \end{cases}$$

Then, we have

$$-\int_{-1}^{1} f(x)\phi'(x)dx = -\int_0^1 \phi'(x)dx = \phi(0) = \delta_0(\phi).$$

Compare this with the definition of weak derivatives, we may regard $\delta_0(x)$ as $f'(x)$ in the sense of distributions.

1.2 Sobolev Spaces

Now suppose given a function $f \in L^p(\Omega)$, we want to solve the partial differential equation

$$\Delta u = f(x)$$

in the sense of weak derivatives. Naturally, we would seek solutions u, such that Δu are in $L^p(\Omega)$. More generally, we would start from the collections of all distributions whose second weak derivatives are in $L^p(\Omega)$.

In a variational approach, as we have seen in the introduction, to seek critical points of the functional

$$\frac{1}{2}\int_\Omega |\bigtriangledown u|^2 dx - \int_\Omega F(x,u)dx,$$

a natural set of functions we start with is the collection of distributions whose first weak derivatives are in $L^2(\Omega)$. More generally, we have

Definition 1.2.1 *The Sobolev space $W^{k,p}(\Omega)$ ($k \geq 0$ and $p \geq 1$) is the collection of all distributions u on Ω such that for all multi-index α with $|\alpha| \leq k$, $D^\alpha u$ can be realized as a L^p function on Ω. Furthermore, $W^{k,p}(\Omega)$ is a Banach space with the norm*

$$||u|| := \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p dx \right)^{1/p}.$$

In the special case when $p = 2$, it is also a Hilbert space and we usually denote it by $H^k(\Omega)$.

Definition 1.2.2 *$W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

Intuitively, $W_0^{k,p}(\Omega)$ is the space of functions whose up to $(k-1)th$ order derivatives vanish on the boundary.

Example 1.2.1 *Let $\Omega = (-1,1)$. Consider the function $f(x) = |x|^\beta$. For $0 < \beta < 1$, it is obviously not differentiable at the origin. However for any $1 \leq p < \dfrac{1}{1-\beta}$, it is in the Sobolev space $W^{1,p}(\Omega)$. More generally, let Ω be an open unit ball centered at the origin in R^n, then the function $|x|^\beta$ is in $W^{1,p}(\Omega)$ if and only if*

$$\beta > 1 - \frac{n}{p}. \tag{1.8}$$

To see this, we first calculate

$$f_{x_i}(x) = \frac{\beta x_i}{|x|^{2-\beta}}, \quad \text{for } x \neq 0,$$

and hence

$$|\bigtriangledown f(x)| = \frac{\beta}{|x|^{1-\beta}}. \tag{1.9}$$

Fix a small $\epsilon > 0$. Then for any $\phi \in D(\Omega)$, by integration by parts, we have

$$\int_{\Omega \setminus B_\epsilon(0)} f_{x_i}(x)\phi(x)dx = -\int_{\Omega \setminus B_\epsilon(0)} f(x)\phi_{x_i}(x)dx + \int_{\partial B_\epsilon(0)} f\phi\nu_i dS.$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is an inward-normal vector on $\partial B_\epsilon(0)$.

Now, under the condition that $\beta > 1 - n/p$, f_{x_i} is in $L^p(\Omega) \subset L^1(\Omega)$, and

$$|\int_{\partial B_\epsilon(0)} f\phi\nu_i dS| \leq C\epsilon^{n-1+\beta} \to 0, \quad as\ \epsilon \to 0.$$

It follows that

$$\int_\Omega |x|^\beta \phi_{x_i}(x)dx = -\int_\Omega \frac{\beta x_i}{|x|^{2-\beta}}\phi(x)dx.$$

Therefore, the weak first partial derivatives of $|x|^\beta$ are

$$\frac{\beta x_i}{|x|^{2-\beta}}\ , \quad i = 1, 2, \cdots, n.$$

Moreover, from (1.9), one can see that $|\bigtriangledown f|$ is in $L^p(\Omega)$ if and only if $\beta > 1 - \frac{n}{p}$.

1.3 Approximation by Smooth Functions

While working in Sobolev spaces, for instance, proving inequalities, it may feel inconvenient and cumbersome to manage the weak derivatives directly based on its definition. To get around, in this section, we will show that any function in a Sobolev space can be approached by a sequence of smooth functions. In other words, the smooth functions are dense in Sobolev spaces. Based on this, when deriving many useful properties of Sobolev spaces, we can just work on smooth functions and then take limits.

At the end of the section, for more convenient application of the approximation results, we introduce an *Operator Completion Theorem*. We also prove an *Extension Theorem*. Both theorems will be used frequently in the next few sections.

The idea in approximation is based on mollifiers. Let

$$j(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

One can verify that

$$j(x) \in C_0^\infty(B_1(0)).$$

Choose the constant c, such that

$$\int_{R^n} j(x)dx = 1.$$

For each $\epsilon > 0$, define

$$j_\epsilon(x) = \frac{1}{\epsilon^n} j(\frac{x}{\epsilon}).$$

Then obviously

$$\int_{R^n} j_\epsilon(x)dx = 1, \quad \forall \epsilon > 0.$$

One can also verify that $j_\epsilon \in C_0^\infty(B_\epsilon(0))$, and

$$\lim_{\epsilon \to 0} j_\epsilon(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0. \end{cases}$$

The above observations suggest that the limit of $j_\epsilon(x)$ may be viewed as a delta function, and from the well-known property of the delta function, we would naturally expect, for any continuous function $f(x)$ and for any point $x \in \Omega$,

$$(J_\epsilon f)(x) := \int_\Omega j_\epsilon(x-y)f(y)dy \to f(x), \quad \text{as } \epsilon \to 0. \tag{1.10}$$

More generally, if $f(x)$ is only in $L^p(\Omega)$, we would expect (1.10) to hold almost everywhere. We call $j_\epsilon(x)$ a mollifier and $(J_\epsilon f)(x)$ the mollification of $f(x)$. We will show that, for each $\epsilon > 0$, $J_\epsilon f$ is a C^∞ function, and as $\epsilon \to 0$,

$$J_\epsilon f \to f \quad \text{in } W^{k,p}.$$

Notice that, actually

$$(J_\epsilon f)(x) = \int_{B_\epsilon(x) \cap \Omega} j_\epsilon(x-y)f(y)dy,$$

hence, in order that $J_\epsilon f(x)$ to approximate $f(x)$ well, we need $B_\epsilon(x)$ to be completely contained in Ω to ensure that

$$\int_{B_\epsilon(x) \cap \Omega} j_\epsilon(x-y)dy = 1$$

(one of the important property of delta function). Equivalently, we need x to be in the interior of Ω. For this reason, we first prove a local approximation theorem.

Theorem 1.3.1 *(Local approximation by smooth functions).*
For any $f \in W^{k,p}(\Omega)$, $J_\epsilon f \in C^\infty(R^n)$ and $J_\epsilon f \to f$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$.

Then, to extend this result to the entire Ω, we will choose infinitely many open sets O_i, $i = 1, 2, \cdots$, each of which has a positive distance to the boundary of Ω, and whose union is the whole Ω. Based on the above theorem, we are able to approximate a $W^{k,p}(\Omega)$ function on each O_i by a sequence of smooth functions. Combining this with a partition of unity, and a cut off function if Ω is unbounded, we will then prove

Theorem 1.3.2 *(Global approximation by smooth functions).*
For any $f \in W^{k,p}(\Omega)$, there exists a sequence of functions $\{f_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$ as $m{\to}\infty$.

Theorem 1.3.3 *(Global approximation by smooth functions up to the boundary).*
Assume that Ω is bounded with C^1 boundary $\partial\Omega$, then for any $f \in W^{k,p}(\Omega)$, there exists a sequence of functions $\{f_m\} \subset C^\infty(\overline{\Omega}) = C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$ as $m{\to}\infty$.

When Ω is the entire space R^n, the approximation by C^∞ or by C_0^∞ functions are essentially the same. We have

Theorem 1.3.4 $W^{k,p}(R^n) = W_0^{k,p}(R^n)$. *In other words, for any $f \in W^{k,p}(R^n)$, there exists a sequence of functions $\{f_m\} \subset C_0^\infty(R^n)$, such that*

$$f_m{\to}f, \quad \textit{as } m{\to}\infty; \quad \textit{in } W^{k,p}(R^n).$$

Proof of Theorem 1.3.1

We prove the theorem in three steps.

In step 1, we show that $J_\epsilon f \in C^\infty(R^n)$ and

$$\|J_\epsilon f\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

From the definition of $J_\epsilon f(x)$, we can see that it is well defined for all $x \in R^n$, and it vanishes if x is of ϵ distance away from Ω. Here and in the following, for simplicity of argument, we extend f to be zero outside of Ω.

In step 2, we prove that, if f is in $L^p(\Omega)$,

$$(J_\epsilon f){\to}f \quad \text{in } L^p_{loc}(\Omega).$$

We first verify this for continuous functions and then approximate L^p functions by continuous functions.

In step 3, we reach the conclusion of the Theorem. For each $f \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$, $D^\alpha f$ is in $L^p(\Omega)$. Then from the result in Step 2, we have

$$J_\epsilon(D^\alpha f){\to}D^\alpha f \quad \text{in } L_{loc}(\Omega).$$

Hence what we need to verify is

$$D^\alpha(J_\epsilon f)(x) = J_\epsilon(D^\alpha f)(x).$$

As the readers will notice, the arguments in the last two steps only work in any compact subset of Ω.

Step 1.

Let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ be the unit vector in the x_i direction.

Fix $\epsilon > 0$ and $x \in R^n$. By the definition of $J_\epsilon f$, we have, for $|h| < \epsilon$,

$$\frac{(J_\epsilon f)(x + he_i) - (J_\epsilon f)(x)}{h} = \int_{B_{2\epsilon}(x)\cap\Omega} \frac{j_\epsilon(x + he_i - y) - j_\epsilon(x - y)}{h} f(y)dy. \tag{1.11}$$

Since as $h \rightarrow 0$,

$$\frac{j_\epsilon(x + he_i - y) - j_\epsilon(x - y)}{h} \rightarrow \frac{\partial j_\epsilon(x - y)}{\partial x_i}$$

uniformly for all $y \in B_{2\epsilon}(x) \cap \Omega$, we can pass the limit through the integral sign in (1.11) to obtain

$$\frac{\partial (J_\epsilon f)(x)}{\partial x_i} = \int_\Omega \frac{\partial j_\epsilon(x - y)}{\partial x_i} f(y)dy.$$

Similarly, we have

$$D^\alpha(J_\epsilon f)(x) = \int_\Omega D_x^\alpha j_\epsilon(x - y) f(y)dy.$$

Noticing that $j_\epsilon(\cdot)$ is infinitely differentiable, we conclude that $J_\epsilon f$ is also infinitely differentiable.

Then we derive

$$\|J_\epsilon f\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}. \tag{1.12}$$

By the Hölder inequality, we have

$$\begin{aligned}
|(J_\epsilon f)(x)| &= |\int_\Omega j_\epsilon^{\frac{p-1}{p}}(x - y) j_\epsilon^{\frac{1}{p}}(x - y) f(y)dy| \\
&\le \left(\int_\Omega j_\epsilon(x - y)dy\right)^{\frac{p-1}{p}} \left(\int_\Omega j_\epsilon(x - y)|f(y)|^p dy\right)^{\frac{1}{p}} \\
&\le \left(\int_{B_\epsilon(x)} j_\epsilon(x - y)|f(y)|^p dy\right)^{\frac{1}{p}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_\Omega |(J_\epsilon f)(x)|^p dx &\le \int_\Omega \left(\int_{B_\epsilon(x)} j_\epsilon(x - y)|f(y)|^p dy\right) dx \\
&\le \int_\Omega |f(y)|^p \left(\int_{B_\epsilon(y)} j_\epsilon(x - y)dx\right) dy \\
&\le \int_\Omega |f(y)|^p dy.
\end{aligned}$$

This verifies (1.12).

Step 2.
We prove that, for any compact subset K of Ω,

$$\|J_\epsilon f - f\|_{L^p(K)} \to 0, \quad \text{as } \epsilon \to 0. \tag{1.13}$$

We first show this for a continuous function f. By writing

$$(J_\epsilon f)(x) - f(x) = \int_{B_\epsilon(0)} j_\epsilon(y)[f(x-y) - f(x)]dy,$$

we have

$$\begin{aligned} |(J_\epsilon f)(x) - f(x)| &\leq \max_{x \in K, |y|<\epsilon} |f(x-y) - f(x)| \int_{B_\epsilon(0)} j_\epsilon(y)dy \\ &\leq \max_{x \in K, |y|<\epsilon} |f(x-y) - f(x)|. \end{aligned}$$

Due to the continuity of f and the compactness of K, the last term in the above inequality tends to zero uniformly as $\epsilon \to 0$. This verifies (1.13) for continuous functions.

For any f in $L^p(\Omega)$, and given any $\delta > 0$, choose a a continuous function g, such that

$$\|f - g\|_{L^p(\Omega)} < \frac{\delta}{3}. \tag{1.14}$$

This can be derived from the well-known fact that any L^p function can be approximated by a simple function of the form $\sum_{j=1}^k a_j \chi_j(x)$, where χ_j is the characteristic function of some measurable set A_j; and each simple function can be approximated by a continuous function.

For the continuous function g, (1.13) infers that for sufficiently small ϵ, we have

$$\|J_\epsilon g - g\|_{L^p(K)} < \frac{\delta}{3}.$$

It follows from this and (1.14) that

$$\begin{aligned} \|J_\epsilon f - f\|_{L^p(K)} &\leq \|J_\epsilon f - J_\epsilon g\|_{L^p(K)} + \|J_\epsilon g - g\|_{L^p(K)} + \|g - f\|_{L^p(K)} \\ &\leq 2\|f - g\|_{L^p(K)} + \|J_\epsilon g - g\|_{L^p(K)} \\ &\leq 2 \cdot \frac{\delta}{3} + \frac{\delta}{3} \\ &= \delta. \end{aligned}$$

This proves (1.13).

Step 3.
Now assume that $f \in W^{k,p}(\Omega)$. Then for any α with $|\alpha| \leq k$, we have $D^\alpha f \in L^p(\Omega)$. We show that

$$D^\alpha(J_\epsilon f) \to D^\alpha f \quad \text{in } L^p(K). \tag{1.15}$$

By the result in Step 2, we have

$$J_\epsilon(D^\alpha f) \to D^\alpha f \quad \text{in } L^p(K).$$

Now what left to verify is, for sufficiently small ϵ,

$$D^\alpha(J_\epsilon f)(x) = J_\epsilon(D^\alpha f)(x), \;\; \forall x \in K.$$

To see this, we fix a point x in K. Choose $\epsilon < \frac{1}{2} dist(K, \partial\Omega)$, then any point y in the ball $B_\epsilon(x)$ is in the interior of Ω. Consequently,

$$\begin{aligned} D^\alpha(J_\epsilon f)(x) &= \int_\Omega D_x^\alpha j_\epsilon(x-y) f(y) dy \\ &= \int_{B_\epsilon(x)} D_x^\alpha j_\epsilon(x-y) f(y) dy \\ &= (-1)^{|\alpha|} \int_{B_\epsilon(x)} D_y^\alpha j_\epsilon(x-y) f(y) dy \\ &= \int_{B_\epsilon(x)} j_\epsilon(x-y) D^\alpha f(y) dy \\ &= \int_\Omega j_\epsilon(x-y) D^\alpha f(y) dy. \end{aligned}$$

Here we have used the fact that $j_\epsilon(x-y)$ and all its derivatives are supported in $B_\epsilon(x) \subset \Omega$.

This completes the proof of the Theorem 1.3.1. □

The Proof of Theorem 1.3.2

Step 1. For a Bounded Region Ω.

Let

$$\Omega_i = \{x \in \Omega \mid dist(x, \partial\Omega) > \frac{1}{i}\}, \quad i = 1, 2, 3, \cdots$$

Write $O_i = \Omega_{i+3} \setminus \bar{\Omega}_{i+1}$. Choose some open set $O_0 \subset\subset \Omega$ so that $\Omega = \cup_{i=0}^\infty O_i$. Choose a smooth partition of unity $\{\eta_i\}_{i=0}^\infty$ associated with the open sets $\{O_i\}_{i=0}^\infty$,

$$\begin{cases} 0 \le \eta_i(x) \le 1, \quad \eta_i \in C_0^\infty(O_i) \\ \sum_{i=0}^\infty \eta_i(x) = 1, \, x \in \Omega. \end{cases}$$

Given any function $f \in W^{k,p}(\Omega)$, obvious $\eta_i f \in W^{k,p}(\Omega)$ and $\text{supp}(\eta_i f) \subset O_i$.

Fix a $\delta > 0$. Choose $\epsilon_i > 0$ so small that $f_i := J_{\epsilon_i}(\eta_i f)$ satisfies

$$\begin{cases} \|f_i - \eta_i f\|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^{i+1}} & i = 0, 1, \cdots \\ supp f_i \subset (\Omega_{i+4} \setminus \bar{\Omega}_i) & i = 1, 2, \cdots \end{cases} \tag{1.16}$$

Set $g = \sum_{i=0}^{\infty} f_i$. Then $g \in C^\infty(\Omega)$, because each f_i is in $C^\infty(\Omega)$, and for each open set $O \subset\subset \Omega$, there are at most finitely many nonzero terms in the sum. To see that $g \in W^{k,p}(\Omega)$, we write

$$g - f = \sum_{i=0}^{\infty} (f_i - \eta_i f).$$

It follows from (1.16) that

$$\sum_{i=0}^{\infty} \|f_i - \eta_i f\|_{W^{k,p}(\Omega)} \leq \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

From here we can see that the series $\sum_{i=0}^{\infty}(f_i - \eta_i f)$ converges in $W^{k,p}(\Omega)$ (see Exercise 1.3.1 below), hence $(g - f) \in W^{k,p}(\Omega)$, and therefore $g \in W^{k,p}(\Omega)$. Moreover, from the above inequality, we have

$$\|g - f\|_{W^{k,p}(\Omega)} \leq \delta.$$

Since $\delta > 0$ is any number, we complete step 1.

Remark 1.3.1 *In the above argument, neither $\sum f_i$ nor $\sum \eta_i f$ converge, but their difference converges. Although each partial sum of $\sum f_i$ is in $C_0^\infty(\Omega)$, the infinite series $\sum f_i$ does not converge to g in $W^{k,p}(\Omega)$. Then how did we prove that $g \in W^{k,p}(\Omega)$? We showed that the difference $(g - f)$ is in $W^{k,p}(\Omega)$.*

Exercise 1.3.1 .

Let X be a Banach space and $v_i \in X$. Show that the series $\sum_i^\infty v_i$ converges in X if $\sum_i^\infty \|v_i\| < \infty$.

Hint: Show that the partial sum is a Cauchy sequence.

Step 2. For Unbounded Region Ω.

Given any $\delta > 0$, since $f \in W^{k,p}(\Omega)$, there exist $R > 0$, such that

$$\|f\|_{W^{k,p}(\Omega \setminus B_{R-2}(0))} \leq \delta. \tag{1.17}$$

Choose a cut off function $\phi \in C^\infty(R^n)$ satisfying

$$\phi(x) = \begin{cases} 1, \, x \in B_{R-2}(0), \\ 0, \, x \in R^n \setminus B_R(0); \end{cases} \quad \text{and } |D^\alpha \phi(x)| \leq 1 \;\; \forall x \in R^n, \forall |\alpha| \leq k.$$

Then by (1.17), it is easy to verify that, there exists a constant C, such that

$$\|\phi f - f\|_{W^{k,p}(\Omega)} \leq C\delta. \tag{1.18}$$

Now in the bounded domain $\Omega \cap B_R(0)$, by the argument in *Step 1*, there is a function $g \in C^\infty(\Omega \cap B_R(0))$, such that

$$\|g - f\|_{W^{k,p}(\Omega \cap B_R(0))} \leq \delta. \tag{1.19}$$

Obviously, the function ϕg is in $C^\infty(\Omega)$, and by (1.18) and (1.19),

$$\begin{aligned}\|\phi g - f\|_{W^{k,p}(\Omega)} &\leq \|\phi g - \phi f\|_{W^{k,p}(\Omega)} + \|\phi f - f\|_{W^{k,p}(\Omega)} \\ &\leq C_1 \|g - f\|_{W^{k,p}(\Omega \cap B_R(0))} + C\delta \\ &\leq (C_1 + C)\delta.\end{aligned}$$

This completes the proof of the Theorem 1.3.2. □

The Proof of Theorem 1.3.3

We will still use the mollifiers to approximate a function. As compared to the proofs of the previous theorem, the main difficulty here is that for a point on the boundary, there is no room to mollify. To circumvent this, we cover Ω with finitely many open sets, and on each set that covers the boundary layer, we will translate the function a little bit inward, so that there is room to mollify within Ω. Then we will again use the partition of unity to complete the proof.

Step 1. Approximating in a Small Set Covering $\partial\Omega$.

Let x^o be any point on $\partial\Omega$. Since $\partial\Omega$ is C^1, we can make a C^1 change of coordinates locally, so that in the new coordinates system $(x_1, \cdots, x_n)$, we can express, for a sufficiently small $r > 0$,

$$B_r(x^o) \cap \Omega = \{x \in B_r(x^o) \mid x_n > \phi(x_1, \cdots x_{n-1})\}$$

with some C^1 function Φ.

Set

$$D = \Omega \cap B_{r/2}(x^o).$$

Shift every point $x \in D$ in x_n direction $a\epsilon$ units, define

$$x^\epsilon = x + a\epsilon e_n.$$

and

$$D^\epsilon = \{x^\epsilon \mid x \in D\}.$$

This D^ϵ is obtained by shifting D toward the inside of Ω by $a\epsilon$ units. Choose a sufficiently large, so that the ball $B_\epsilon(x)$ lies in $\Omega \cap B_r(x^o)$ for all $x \in D^\epsilon$ and for all small $\epsilon > 0$. Now, there is room to mollify a given $W^{k,p}$ function f on D^ϵ within Ω. More precisely, we first translate f a distance ϵ in the x_n direction to become $f^\epsilon(x) = f(x^\epsilon)$, then mollify it. We claim that

$$J_\epsilon f^\epsilon \rightarrow f \quad \text{ in } W^{k,p}(D).$$

Actually, for any multi-index $|\alpha| \leq k$, we have

$$\|D^\alpha(J_\epsilon f^\epsilon) - D^\alpha f\|_{L^p(D)} \leq \|D^\alpha(J_\epsilon f^\epsilon) - D^\alpha f^\epsilon\|_{L^p(D)} + \|D^\alpha f^\epsilon - D^\alpha f\|_{L^p(D)}.$$

A similar argument as in the proof of Theorem 1.3.1 implies that the first term on the right hand side goes to zero as $\epsilon \to 0$; while the second term also vanishes in the process due to the continuity of the translation in the L^p norm.

Step 2. Applying the Partition of Unity.

Since $\partial\Omega$ is compact, we can find finitely many such sets D, call them D_i, $i = 1, 2, \cdots, N$, the union of which covers $\partial\Omega$. Given $\delta > 0$, from the argument in *Step 1*, for each D_i, there exists $g_i \in C^\infty(\bar{D}_i)$, such that

$$\|g_i - f\|_{W^{k,p}(D_i)} \leq \delta. \tag{1.20}$$

Choose an open set $D_0 \subset\subset \Omega$ such that $\Omega \subset \cup_{i=0}^N D_i$, and select a function $g_0 \in C^\infty(\bar{D}_0)$ such that

$$\|g_0 - f\|_{W^{k,p}(D_0)} \leq \delta. \tag{1.21}$$

Let $\{\eta_i\}$ be a smooth partition of unity subordinated to the open sets $\{D_i\}_{i=0}^N$. Define

$$g = \sum_{i=0}^N \eta_i g_i.$$

Then obviously $g \in C^\infty(\bar{\Omega})$, and $f = \sum_{i=0}^N \eta_i f$. Similar to the proof of Theorem 1.3.1, it follows from (1.20) and (1.21) that, for each $|\alpha| \leq k$

$$\begin{aligned} \|D^\alpha g - D^\alpha f\|_{L^p(\Omega)} &\leq \sum_{i=0}^N \|D^\alpha(\eta_i g_i) - D^\alpha(\eta_i f)\|_{L^p(D_i)} \\ &\leq C \sum_{i=0}^N \|g_i - f\|_{W^{k,p}(D_i)} = C(N+1)\delta. \end{aligned}$$

This completes the proof of the Theorem 1.3.3. □

The Proof of Theorem 1.3.4

Let $\phi(r)$ be a C_0^∞ cut off function such that

$$\phi(r) = \begin{cases} 1 \,, & \text{for } 0 \leq r \leq 1; \\ \text{between 0 and 1} \,, & \text{for } 1 < r < 2; \\ 0 \,, & \text{for } r \geq 2. \end{cases}$$

Then by a direct computation, one can show that

$$\|\phi(\frac{|x|}{R}) f(x) - f(x)\|_{W^{k,p}(R^n)} \to 0, \quad \text{as } R \to \infty. \tag{1.22}$$

Thus, there exists a sequence of numbers $\{R_m\}$ with $R_m \to \infty$, such that for

$$g_m(x) := \phi(\frac{|x|}{R_m})f(x),$$

we have

$$\|g_m - f\|_{W^{k,p}(R^n)} \leq \frac{1}{m}. \tag{1.23}$$

On the other hand, from the Approximation Theorem 1.3.3, for each fixed m,

$$J_\epsilon(g_m)\to g_m, \quad \text{as } \epsilon\to 0, \quad \text{in } W^{k,p}(R^n).$$

Hence there exist ϵ_m, such that for

$$f_m := J_{\epsilon_m}(g_m),$$

we have

$$\|f_m - g_m\|_{W^{k,p}(R^n)} \leq \frac{1}{m}. \tag{1.24}$$

Obviously, each function f_m is in $C_0^\infty(R^n)$, and by (1.23) and (1.24),

$$\|f_m - f\|_{W^{k,p}(R^n)} \leq \|f_m - g_m\|_{W^{k,p}(R^n)} + \|g_m - f\|_{W^{k,p}(R^n)} \leq \frac{2}{m}\to 0, \text{ as } m\to\infty.$$

This completes the proof of Theorem 1.3.4. □

Now we have proved all the four Approximation Theorems, which show that smooth functions are dense in Sobolev spaces $W^{k,p}$. In other words, $W^{k,p}$ is the completion of C^k under the norm $\|\cdot\|_{W^{k,p}}$. Later, particularly in the next three sections, when we derive various inequalities in Sobolev spaces, we can just first work on smooth functions and then extend them to the whole Sobolev spaces. In order to make such extensions more conveniently, without going through approximation process in each particular space, we prove the following Operator Completion Theorem in general Banach spaces.

Theorem 1.3.5 *Let D be a dense linear subspace of a normed space X. Let Y be a Banach space. Assume*

$$T : D \mapsto Y$$

is a bounded linear map. Then there exists an extension $\bar{T}$ of T from D to the whole space X, such that $\bar{T}$ is a bounded linear operator from X to Y,

$$\|\bar{T}\| = \|T\| \quad \text{and} \quad \bar{T}x = Tx \ \ \forall\, x \in D.$$

Proof. Given any element $x_o \in X$, since D is dense in X, there exists a sequence $\{x_i\} \in D$, such that

$$x_i\to x_o, \quad \text{as } i\to\infty.$$

It follows that

$$\|Tx_i - Tx_j\| \le \|T\|\|x_i - x_j\| \to 0, \quad \text{as } i,j \to \infty.$$

This implies that $\{Tx_i\}$ is a Cauchy sequence in Y. Since Y is a Banach space, $\{Tx_i\}$ converges to some element y_o in Y. Let

$$\bar{T}x_o = y_o. \tag{1.25}$$

To see that (1.25) is well defined, suppose there is another sequence $\{x_i'\}$ that converges to x_o in X and $Tx_i' \to y_1$ in Y. Then

$$\|y_1 - y_o\| = \lim_{i\to\infty} \|Tx_i' - Tx_i\| \le C\overline{\lim}_{i\to\infty}\|x_i' - x_i\| = 0.$$

Now, for $x \in D$, define $\bar{T}x = Tx$; and for other $x \in X$, define $\bar{T}x$ by (1.25). Obviously, $\bar{T}$ is a linear operator. Moreover

$$\|\bar{T}x_o\| = \lim_{i\to\infty} \|Tx_i\| \le \lim_{i\to\infty} \|T\|\|x_i\| = \|T\|\|x_o\|.$$

Hence $\bar{T}$ is also bounded. This completes the proof. □

As a direct application of this Operator Completion Theorem, we prove the following Extension Theorem.

Theorem 1.3.6 *Assume that Ω is bounded and $\partial\Omega$ is C^k, then for any open set $O \supset \overline{\Omega}$, there exists a bounded linear operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(O)$ such that $Eu = u$ a.e. in Ω.*

Proof. To define the extension operator E, we first work on functions $u \in C^k(\bar{\Omega})$. Then we can apply the Operator Completion Theorem to extend E to $W^{k,p}(\Omega)$, since $C^k(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ (see Theorem 1.3.3). From this density, it is easy to show that, for $u \in W^{k,p}(\Omega)$,

$$E(u)(x) = u(x) \quad \text{almost everywhere on } \Omega$$

based on

$$E(u)(x) = u(x) \quad \text{on } \bar{\Omega} \ \ \forall\, u \in C^k(\bar{\Omega}).$$

Now we prove the theorem for $u \in C^k(\bar{\Omega})$. We define $E(u)$ in the following two steps.

Step 1. The special case when Ω is a half ball

$$B_r^+(0) := \{x = (x', x_n) \in R^n \mid |x| < 1, \ x_n > 0\}.$$

Assume $u \in C^k(\overline{B_r^+(0)})$. We extend u to be a C^k function on the whole ball $\overline{B_r(0)}$. Define

$$\bar{u}(x', x_n) = \begin{cases} u(x', x_n)\,, & x_n \ge 0 \\ \sum_{i=0}^k a_i\, u(x', -\lambda_i x_n)\,, & x_n < 0. \end{cases}$$

To guarantee that all the partial derivatives $\dfrac{\partial^j \bar{u}(x', x_n)}{\partial x_n^j}$ up to the order k are continuous across the hyper plane $x_n = 0$, we first pick λ_i, such that

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < 1.$$

Then solve the algebraic system

$$\sum_{i=0}^{k} a_i \lambda_i^j = 1 \ , \ \ j = 0, 1, \cdots, k,$$

to determine $a_0, a_1, \cdots, a_k$. Since the coefficient determinant $\det(\lambda_i^j)$ is not zero, and hence the system has a unique solution. One can easily verify that, for such λ_i and a_i, the extended function $\bar{u}$ is in $C^k(\overline{B_r(0)})$.

Step 2. Reduce the general domains to the case of half ball covered in Step 1 via domain transformations and a partition of unity.

Let O be an open set containing $\bar{\Omega}$. Given any $x^o \in \partial\Omega$, there exists a neighborhood U of x^o and a C^k transformation $\phi : U \to R^n$ which satisfies that $\phi(x^o) = 0$, $\phi(U \cap \partial\Omega)$ lies on the hyper plane $x_n = 0$, and $\phi(U \cap \Omega)$ is contained in R^n_+. Then there exists an $r_o > 0$, such that $B_{r_o}(0) \subset\subset \phi(U)$, $D_{x^o} := \phi^{-1}(B_{r_o}(0))$ is an open set containing x^o, and $\phi \in C^k(\overline{D_{x^o}})$. Choose r_o sufficiently small so that $D_{x^o} \subset O$. All such D_{x^o} and Ω forms an open covering of $\bar{\Omega}$, hence there exist a finite sub-covering

$$D_0 := \Omega, \ D_1, \ \cdots, \ D_m$$

and a corresponding partition of unity

$$\eta_0, \eta_1, \cdots, \eta_m,$$

such that

$$\eta_i \in C_0^\infty(D_i) \ \ i = 0, 1, \cdots, m$$

and

$$\sum_{i=0}^{m} \eta_i(x) = 1 \ \ \forall\, x \in \Omega.$$

Let ϕ_i be the mapping associated with D_i as described in *Step 1.* And let $\tilde{u}_i(y) = u(\phi_i^{-1}(y)$ be the function defined on the half ball

$$\overline{B^+_{r_i}(0)} = \phi_i(\overline{D_i} \cap \bar{\Omega}) \ , \ \ i = 1, 2, \cdots m.$$

From *Step 1,* each $\tilde{u}_i(y)$ can be extended as a C^k function $\bar{u}_i(y)$ onto the whole ball $\overline{B_{r_i}(0)}$.

Now, we can define the extension of u from Ω to its neighborhood O as

$$E(u) = \sum_{i=1}^{m} \eta_i(x) \bar{u}_i(\phi_i(x)) + \eta_0 u(x).$$

Obviously, $E(u) \in C_0^\infty(O)$, and

$$\|E(u)\|_{W^{k,p}(O)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Moreover, for any $x \in \bar{\Omega}$, we have

$$\begin{aligned} E(u)(x) &= \sum_{i=1}^m \eta_i(x)\bar{u}_i(\phi_i(x)) + \eta_0(x)u(x) = \sum_{i=1}^m \eta_i(x)\tilde{u}_i(\phi_i(x)) + \eta_0(x)u(x) \\ &= \sum_{i=1}^m \eta_i(x)u(\phi_i^{-1}(\phi_i(x))) + \eta_0(x)u(x) = \sum_{i=1}^m \eta_i(x)u(x) + \eta_0(x)u(x) \\ &= \left(\sum_{i=0}^m \eta_i(x)\right) u(x) = u(x). \end{aligned}$$

This completes the proof of the Extension Theorem. □

Remark 1.3.2 *Notice that the Extension Theorem actually implies an improved version of the Approximation Theorem under stronger assumption on $\partial\Omega$ (be C^k instead of C^1). From the Extension Theorem, one can derive immediately*

Corollary 1.3.1 *Assume that Ω is bounded and $\partial\Omega$ is C^k. Let O be an open set containing $\bar{\Omega}$. Let*

$$O_\epsilon := \{x \in R^n \mid dist(x, O) \leq \epsilon\}.$$

Then the linear operator

$$J_\epsilon(E(u)) : W^{k,p}(\Omega) {\rightarrow} C_0^\infty(O_\epsilon)$$

is bounded, and for each $u \in W^{k,p}(\Omega)$, we have

$$\|J_\epsilon(E(u)) - u\|_{W^{k,p}(\Omega)} {\rightarrow} 0 , \quad as\ \epsilon {\rightarrow} 0.$$

Here, as compare to the previous approximation theorems, the improvement is that one can write out the explicit form of the approximation.

1.4 Sobolev Embeddings

When we seek weak solutions of partial differential equations, we start with functions in a Sobolev space $W^{k,p}$. Then it is natural that we would like to know whether the functions in this space also automatically belong to some other spaces. The following theorem answers the question and at meantime provides inequalities among the relevant norms.

Theorem 1.4.1 *(General Sobolev inequalities).*

Assume Ω is bounded and has a C^1 boundary. Let $u \in W^{k,p}(\Omega)$.

(i) If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and there exists a constant C such that

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\Omega)$, and there exists a constant C, such that

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}$$

where

$$\gamma = \begin{cases} 1 + \left[\frac{n}{p}\right] - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer} . \end{cases}$$

Here, $[b]$ is the integer part of the number b.

The proof of the Theorem is based upon several simpler theorems. We first consider the functions in $W^{1,p}(\Omega)$. From the definition, apparently these functions belong to $L^q(\Omega)$ for $1 \leq q \leq p$. Naturally, one would expect more, and what is more meaningful is to find out how large this q can be. And to control the L^q norm for larger q by $W^{1,p}$ norm–the norm of the derivatives

$$\|Du\|_{L^p(\Omega)} = \left(\int_\Omega |Du|^p dx\right)^{\frac{1}{p}}$$

–would suppose to be more useful in practice.

For simplicity, we start with the smooth functions with compact supports in R^n. We would like to know for what value of q, can we establish the inequality

$$\|u\|_{L^q(R^n)} \leq C\|Du\|_{L^p(R^n)}, \tag{1.26}$$

with constant C independent of $u \in C_0^\infty(R^n)$. Now suppose (1.26) holds. Then it must also be true for the re-scaled function of u:

$$u_\lambda(x) = u(\lambda x),$$

that is

$$\|u_\lambda\|_{L^q(R^n)} \leq C\|Du_\lambda\|_{L^p(R^n)}. \tag{1.27}$$

By substitution, we have

$$\int_{R^n} |u_\lambda(x)|^q dx = \frac{1}{\lambda^n}\int_{R^n} |u(y)|^q dy, \text{ and}$$
$$\int_{R^n} |Du_\lambda|^p dx = \frac{\lambda^p}{\lambda^n}\int_{R^n} |Du(y)|^p dy.$$

It follows from (1.27) that

$$\|u\|_{L^q(R^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(R^n)}.$$

Therefore, in order that C to be independent of u, it is necessarily that the power of λ here be zero, that is

$$q = \frac{np}{n-p}.$$

It turns out that this condition is also sufficient, as will be stated in the following theorem. Here for q to be positive, we must require $p < n$.

Theorem 1.4.2 *(Gagliardo-Nirenberg-Sobolev inequality).*
Assume that $1 \leq p < n$. Then there exists a constant $C = C(p,n)$, such that

$$\|u\|_{L^{p^*}(R^n)} \leq C\|Du\|_{L^p(R^n)}, \quad u \in C_0^1(R^n) \tag{1.28}$$

where $p^* = \frac{np}{n-p}$.

Since any function in $W^{1,p}(R^n)$ can be approached by a sequence of functions in $C_0^1(R^n)$, we derive immediately

Corollary 1.4.1 *Inequality (1.28) holds for all functions u in $W^{1,p}(R^n)$.*

For functions in $W^{1,p}(\Omega)$, we can extended them to be $W^{1,p}(R^n)$ functions by Extension Theorem and arrive at

Theorem 1.4.3 *Assume that Ω is a bounded, open subset of R^n with C^1 boundary. Suppose that $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. Then u is in $L^{p*}(\Omega)$ and there exists a constant $C = C(p,n,\Omega)$, such that*

$$\|u\|_{L^{p*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}. \tag{1.29}$$

In the limiting case as $p{\rightarrow}n$, $p* := \frac{np}{n-p}{\rightarrow}\infty$. Then one may suspect that $u \in L^\infty$ when $p = n$. Unfortunately, this is true only in dimension one. For $n = 1$, from

$$u(x) = \int_{-\infty}^{x} u'(x)dx,$$

we derive immediately that

$$|u(x)| \leq \int_{-\infty}^{\infty} |u'(x)|dx.$$

However, for $n > 1$, it is false. One counter example is $u = \log\log(1 + \frac{1}{|x|})$ on $\Omega = B_1(0)$. It belongs to $W^{1,n}(\Omega)$, but not to $L^\infty(\Omega)$. This is a more delicate situation, and we will deal with it later.

Naturally, for $p > n$, one would expect $W^{1,p}$ to embed into better spaces. To get some rough idea what these spaces might be, let us first consider the simplest case when $n = 1$ and $p > 1$. Obviously, for any $x, y \in R^1$ with $x < y$, we have

$$u(y) - u(x) = \int_x^y u'(t)dt,$$

and consequently, by Hölder inequality,

$$|u(y) - u(x)| \leq \int_x^y |u'(t)|dt \leq \left(\int_x^y |u'(t)|^p dt\right)^{\frac{1}{p}} \cdot \left(\int_x^y dt\right)^{1-\frac{1}{p}}.$$

It follows that

$$\frac{|u(y) - u(x)|}{|y - x|^{1-\frac{1}{p}}} \leq \left(\int_{-\infty}^{\infty} |u'(t)|^p dt\right)^{\frac{1}{p}}.$$

Take the supremum over all pairs x, y in R^1, the left hand side of the above inequality is the norm in the Hölder space $C^{0,\gamma}(R^1)$ with $\gamma = 1 - \frac{1}{p}$. This is indeed true in general, and we have

Theorem 1.4.4 *(Morrey's inequality).*
Assume $n < p \leq \infty$. Then there exists a constant $C = C(p, n)$ such that

$$\|u\|_{C^{0,\gamma}(R^n)} \leq C\|u\|_{W^{1,p}(R^n)}, \quad u \in C^1(R^n)$$

where $\gamma = 1 - \frac{n}{p}$.

To establish an inequality like (1.28), we follow two basic principles.

First, we consider the special case where $\Omega = R^n$. Instead of dealing with functions in $W^{k,p}$, we reduce the proof to functions with enough smoothness (which is just Theorem 1.4.2).

Secondly, we deal with general domains by extending functions $u \in W^{1,p}(\Omega)$ to $W^{1,p}(R^n)$ via the Extension Theorem.

Here, one sees that Theorem 1.4.2 is the 'key' and inequality (1.27) there provides the foundation for the proof of the Sobolev embedding. Often, the proof of (1.27) is called a 'hard analysis', and the steps leading from (1.27) to (1.28) are called 'soft analyseses'. In the following, we will show the 'soft' parts first and then the 'hard' ones. First, we will assume that Theorem 1.4.2 is true and derive Corollary 1.4.1 and Theorem 1.4.3. Then, we will prove Theorem 1.4.2.

The Proof of Corollary 1.4.1.

Given any function $u \in W^{1,p}(R^n)$, by the Approximation Theorem, there exists a sequence $\{u_k\} \subset C_0^\infty(R^n)$ such that $\|u - u_k\|_{W^{1,p}(R^n)} \to 0$ as $k \to \infty$.

Applying Theorem 1.4.2, we obtain

$$\|u_i - u_j\|_{L^{p^*}(R^n)} \leq C\|u_i - u_j\|_{W^{1,p}(R^n)} \to 0, \text{ as } i, j \to \infty.$$

Thus $\{u_k\}$ also converges to u in $L^{p^*}(R^n)$. Consequently, we arrive at

$$\|u\|_{L^{p^*}(R^n)} = \lim \|u_k\|_{L^{p^*}(R^n)} \leq \lim C\|u_k\|_{W^{1,p}(R^n)} = C\|u\|_{W^{1,p}(R^n)}.$$

This completes the proof of the Corollary. □

The Proof of Theorem 1.4.3.

Now for functions in $W^{1,p}(\Omega)$, to apply inequality (1.27), we first extend them to be functions with compact supports in R^n. More precisely, let O be an open set that covers Ω, by the Extension Theorem (Theorem 1.3.6), for every u in $W^{1,p}(\Omega)$, there exists a function $\tilde{u}$ in $W_0^{1,p}(O)$, such that

$$\tilde{u} = u, \quad \text{almost everywhere in } \Omega;$$

moreover, there exists a constant $C_1 = C_1(p, n, \Omega, O)$, such that

$$\|\tilde{u}\|_{W^{1,p}(O)} \leq C_1\|u\|_{W^{1,p}(\Omega)}. \tag{1.30}$$

Now we can apply the Gagliardo-Nirenberg-Sobolev inequality to $\tilde{u}$ to derive

$$\|u\|_{L^{p^*}(\Omega)} \leq \|\tilde{u}\|_{L^{p^*}(O)} \leq C\|\tilde{u}\|_{W^{1,p}(O)} \leq CC_1\|u\|_{W^{1,p}(\Omega)}.$$

This completes the proof of the Theorem. □

The Proof of Theorem 1.4.2.

We first establish the inequality for $p = 1$, i.e. we prove

$$\left(\int_{R^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \int_{R^n} |Du| dx. \tag{1.31}$$

Then we will apply (1.31) to $|u|^\gamma$ for a properly chosen $\gamma > 1$ to extend the inequality to the case when $p > 1$.

We need

Lemma 1.4.1 *(General Hölder Inequality) Assume that*

$$u_i \in L^{p_i}(\Omega) \textit{ for } i = 1, 2, \cdots, m$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$

Then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_i^m \left(\int_\Omega |u_i|^{p_i} dx\right)^{\frac{1}{p_i}}. \tag{1.32}$$

The proof can be obtained by applying induction to the usual Hölder inequality for two functions.

Now we are ready to prove the Theorem.

Step 1. The case $p = 1$.

To better illustrate the idea, we first derive inequality (1.31) for $n = 2$, that is, we prove

$$\int_{R^2} |u(x)|^2 dx \leq \left(\int_{R^2} |Du| dx \right)^2 . \tag{1.33}$$

Since u has a compact support, we have

$$u(x) = \int_{-\infty}^{x_1} \frac{\partial u}{\partial y_1}(y_1, x_2) dy_1 .$$

It follows that

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(y_1, x_2)| dy_1 .$$

Similarly, we have

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, y_2)| dy_2 .$$

The above two inequalities together imply that

$$|u(x)|^2 \leq \int_{-\infty}^{\infty} |Du(y_1, x_2)| dy_1 \cdot \int_{-\infty}^{\infty} |Du(x_1, y_2)| dy_2 .$$

Now, integrate both sides of the above inequality with respect to x_1 and x_2 from $-\infty$ to ∞, we arrive at (1.33).

Then we deal with the general situation when $n > 2$. We write

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial y_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i , \quad i = 1, 2, \cdots n .$$

Consequently,

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots, x_n)| dy_i , \quad i = 1, 2, \cdots n .$$

And it follows that

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots, x_n)| dy_i \right)^{\frac{1}{n-1}} .$$

Integrating both sides with respect to x_1 and applying the general Hölder inequality (1.32), we obtain

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$\leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}} .$$

Then integrate the above inequality with respect to x_2. We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \leq$$
$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\left\{\left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}}\cdot\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i\right)^{\frac{1}{n-1}}\right\} d$$

Again, applying the General Hölder Inequality, we arrive at

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \leq$$
$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dy_1dx_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\right)^{\frac{1}{n-1}}\times$$
$$\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dx_2dy_i\right)^{\frac{1}{n-1}}.$$

Continuing this way by integrating with respect to $x_3, \cdots, x_{n-1}$, we deduce

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1\cdots dx_{n-1} \leq$$
$$\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|dy_1dx_2\cdots dx_{n-1}\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|dx_1dy_2dx_3\cdots dx_{n-1}\right)^{\frac{1}{n-1}}\cdots\times$$
$$\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|dx_1\cdots dx_{n-2}dy_{n-1}\right)^{\frac{1}{n-1}}\left(\int_{R^n}|Du|dx\right)^{\frac{1}{n-1}}.$$

Finally, integrating both sides with respect to x_n and applying the general Hölder inequality, we obtain

$$\int_{R^n}|u|^{\frac{n}{n-1}}dx \leq \left(\int_{R^n}|Du|dx\right)^{\frac{n}{n-1}}.$$

This verifies (1.31).

Exercise 1.4.1 *Write your own proof with all details for the cases* $n = 3$ *and* $n = 4$.

Step 2. The Case $p > 1$.

Applying (1.31) to the function $|u|^\gamma$ with $\gamma > 1$ to be chosen later, and by the Hölder inequality, we have

$$\left(\int_{R^n}|u|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{n-1}{n}} \leq \int_{R^n}|D(|u|^\gamma)|dx = \gamma\int_{R^n}|u|^{\gamma-1}|Du|dx$$
$$\leq \gamma\left(\int_{R^n}|u|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{(\gamma-1)(n-1)}{\gamma n}}\left(\int_{R^n}|Du|^{\frac{\gamma n}{\gamma+n-1}}dx\right)^{\frac{\gamma+n-1}{\gamma n}}.$$

It follows that

$$\left(\int_{R^n} |u|^{\frac{\gamma n}{n-1}} dx\right)^{\frac{n-1}{\gamma n}} \leq \gamma \left(\int_{R^n} |Du|^{\frac{\gamma n}{\gamma+n-1}} dx\right)^{\frac{\gamma+n-1}{\gamma n}}.$$

Now choose γ, so that $\dfrac{\gamma n}{\gamma+n-1} = p$, that is

$$\gamma = \frac{p(n-1)}{n-p},$$

we obtain

$$\left(\int_{R^n} |u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leq \gamma \left(\int_{R^n} |Du|^p dx\right)^{\frac{1}{p}}.$$

This completes the proof of the Theorem. □

The Proof of Theorem 1.4.4 (Morrey's inequality).
We will establish two inequalities

$$\sup_{R^n} |u| \leq C\|u\|_{W^{1,p}(R^n)}, \tag{1.34}$$

and

$$\sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\|Du\|_{L^p(R^n)}. \tag{1.35}$$

Both of them can be derived from the following

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)-u(x)|dy \leq C\int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}dy, \tag{1.36}$$

where $|B_r(x)|$ is the volume of $B_r(x)$. We will carry the proof out in three steps. In *Step 1*, we prove (1.36), and in *Step 2* and *Step 3*, we verify (1.34) and (1.35), respectively.

Step 1. We start from

$$u(y)-u(x) = \int_0^1 \frac{d}{dt}u(x+t(y-x))dt = \int_0^s Du(x+\tau\omega)\cdot\omega d\tau,$$

where

$$\omega = \frac{y-x}{|x-y|}, \; s = |x-y|, \;\; \text{and hence } y = x+s\omega.$$

It follows that

$$|u(x+s\omega)-u(x)| \leq \int_0^s |Du(x+\tau\omega)|d\tau.$$

Integrating both sides with respect to ω on the unit sphere $\partial B_1(0)$, then converting the integral on the right hand side from polar to rectangular coordinates, we obtain

$$\int_{\partial B_1(0)} |u(x+s\omega)-u(x)|d\sigma \leq \int_0^s \int_{\partial B_1(0)} |Du(x+\tau\omega)|d\sigma d\tau$$
$$= \int_0^s \int_{\partial B_1(0)} |Du(x+\tau\omega)|\frac{\tau^{n-1}}{\tau^{n-1}} d\sigma d\tau$$
$$= \int_{B_s(x)} \frac{|Du(z)|}{|x-z|^{n-1}} dz.$$

Multiplying both sides by s^{n-1}, integrating with respect to s from 0 to r, and taking into account that the integrand on the right hand side is non-negative, we arrive at

$$\int_{B_r(x)} |u(y)-u(x)|dy \leq \frac{r^n}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy.$$

This verifies (1.36).

Step 2. For each fixed $x \in R^n$, we have

$$|u(x)| = \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(x)|dy$$
$$\leq \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(x)-u(y)|dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)|dy$$
$$= I_1 + I_2. \tag{1.37}$$

By (1.36) and the Hölder inequality, we deduce

$$I_1 \leq C \int_{B_1(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy$$
$$\leq C \left(\int_{R^n} |Du|^p dy \right)^{1/p} \left(\int_{B_1(x)} \frac{dy}{|x-y|^{\frac{(n-1)p}{p-1}}} dy \right)^{\frac{p-1}{p}}$$
$$\leq C_1 \left(\int_{R^n} |Du|^p dy \right)^{1/p}. \tag{1.38}$$

Here we have used the condition that $p > n$, so that $\frac{(n-1)p}{p-1} < n$, and hence the integral

$$\int_{B_1(x)} \frac{dy}{|x-y|^{\frac{(n-1)p}{p-1}}}$$

is finite.

Also it is obvious that

$$I_2 \leq C\|u\|_{L^p(R^n)}. \tag{1.39}$$

Now (1.34) is an immediate consequence of (1.37), (1.38), and (1.39).

Step 3. For any pair of fixed points x and y in R^n, let $r = |x-y|$ and $D = B_r(x) \cap B_r(y)$. Then

$$|u(x)-u(y)| \leq \frac{1}{|D|}\int_D |u(x)-u(z)|dz + \frac{1}{|D|}\int_D |u(z)-u(y)|dz. \qquad (1.40)$$

Again by (1.36), we have

$$\begin{aligned} \frac{1}{|D|}\int_D |u(x)-u(z)|dz &\leq \frac{C}{|B_r(x)|}\int_{B_r(x)} |u(x)-u(z)|dz \\ &\leq C\left(\int_{R^n} |Du|^p dz\right)^{1/p} \left(\int_{B_r(x)} \frac{dz}{|x-z|^{\frac{(n-1)p}{p-1}}}\right)^{\frac{p-1}{p}} \\ &= Cr^{1-n/p}\|Du\|_{L^p(R^n)}. \qquad (1.41) \end{aligned}$$

Similarly,

$$\frac{1}{|D|}\int_D |u(z)-u(y)|dz \leq Cr^{1-n/p}\|Du\|_{L^p(R^n)}. \qquad (1.42)$$

Now (1.40), (1.41), and (1.42) yield

$$|u(x)-u(y)| \leq C|x-y|^{1-n/p}\|Du\|_{L^p(R^n)}.$$

This implies (1.35) and thus completes the proof of the Theorem. □

The Proof of Theorem 1.4.1 (the General Sobolev Inequality).

(i) Assume that $u \in W^{k,p}(\Omega)$ with $k < \frac{n}{p}$. We want to show that $u \in L^q(\Omega)$ and

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}. \qquad (1.43)$$

with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. This can be done by applying Theorem 1.4.3 successively on the integer k. Again denote $p^* = \frac{np}{n-p}$, then $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. For $|\alpha| \leq k-1$, $D^\alpha u \in W^{1,p}(\Omega)$. By Theorem 1.4.3, we have $D^\alpha u \in L^{p^*}(\Omega)$, thus $u \in W^{k-1,p^*}(\Omega)$ and

$$\|u\|_{W^{k-1,p^*}(\Omega)} \leq C_1\|u\|_{W^{k,p}(\Omega)}.$$

Applying Theorem 1.4.3 again to $W^{k-1,p^*}(\Omega)$, we have $u \in W^{k-2,p^{**}}(\Omega)$ and

$$\|u\|_{W^{k-2,p^{**}}(\Omega)} \leq C_2\|u\|_{W^{k-1,p^*}(\Omega)} \leq C_2C_1\|u\|_{W^{k,p}(\Omega)},$$

where $p^{**} = \frac{np*}{n-p^*}$, or

$$\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = (\frac{1}{p} - \frac{1}{n}) - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

Continuing this way k times, we arrive at (1.43).

(ii) Now assume that $k > \frac{n}{p}$. Recall that in the Morrey's inequality

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{1.44}$$

we require that $p > n$. However, in our situation, this condition is not necessarily met. To remedy this, we can use the result in the previous step. We can first decrease the order of differentiation to increase the power of summability. More precisely, we will try to find a smallest integer m, such that

$$W^{k,p}(\Omega) \hookrightarrow W^{k-m,q}(\Omega),$$

with $q > n$. That is, we want

$$q = \frac{np}{n - mp} > n,$$

and equivalently,

$$m > \frac{n}{p} - 1.$$

Obviously, the smallest such integer m is

$$m = \left[\frac{n}{p}\right].$$

For this choice of m, we can apply Morrey's inequality (1.44) to $D^\alpha u$, with any $|\alpha| \leq k - m - 1$ to obtain

$$\|D^\alpha u\|_{C^{0,\gamma}(\Omega)} \leq C_1\|u\|_{W^{k-m,q}(\Omega)} \leq C_1 C_2 \|u\|_{W^{k,p}(\Omega)}.$$

Or equivalently,

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Here, when $\frac{n}{p}$ is not an integer,

$$\gamma = 1 - \frac{n}{q} = 1 + \left[\frac{n}{p}\right] - \frac{n}{p}.$$

While $\frac{n}{p}$ is an integer, we have $m = \frac{n}{p}$, and in this case, q can be any number $> n$, which implies that γ can be any positive number < 1.

This completes the proof of the Theorem. □

1.5 Compact Embedding

In the previous section, we proved that $W^{1,p}(\Omega)$ is embedded into $L^{p^*}(\Omega)$ with $p^* = \frac{np}{n-p}$. Obviously, for $q < p^*$, the embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ is still true, if the region Ω is bounded. Actually, due to the strict inequality on the exponent, one can expect more, as stated below.

Theorem 1.5.1 *(Rellich-Kondrachov Compact Embedding).*

Assume that Ω is a bounded open subset in R^n with C^1 boundary $\partial\Omega$. Suppose $1 \leq p < n$. Then for each $1 \leq q < p^*$, $W^{1,p}(\Omega)$ is compactly imbedded into $L^q(\Omega)$:

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega),$$

in the sense that

i) there is a constant C, such that

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \ , \ \ \forall\, u \in W^{1,p}(\Omega); \tag{1.45}$$

and

ii) every bounded sequence in $W^{1,p}(\Omega)$ possesses a convergent subsequence in $L^q(\Omega)$.

Proof. The first part–inequality (1.45)– is included in the general Embedding Theorem. What we need to show is that if $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$, then it possesses a convergent subsequence in $L^q(\Omega)$. This can be derived immediately from the following

Lemma 1.5.1 *Every bounded sequence in $W^{1,1}(\Omega)$ possesses a convergent subsequence in $L^1(\Omega)$.*

We postpone the proof of the Lemma for a moment and see how it implies the Theorem.

In fact, assume that $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Then there exists a subsequence (still denoted by $\{u_k\}$) which converges weakly to an element u_o in $W^{1,p}(\Omega)$. By the Sobolev embedding, $\{u_k\}$ is bounded in $L^{p*}(\Omega)$. On the other hand, it is also bounded in $W^{1,1}(\Omega)$, since $p \geq 1$ and Ω is bounded. Now, by Lemma 1.5.1, there is a subsequence (still denoted by $\{u_k\}$) that converges strongly to u_o in $L^1(\Omega)$. Applying the Hölder inequality

$$\|u_k - u_o\|_{L^q(\Omega)} \leq \|u_k - u_o\|^{\theta}_{L^1(\Omega)} \|u_k - u_o\|^{1-\theta}_{L^{p*}(\Omega)},$$

we conclude immediately that $\{u_k\}$ converges strongly to u_o in $L^q(\Omega)$. This proves the Theorem.

We now come back to prove the Lemma. Let $\{u_k\}$ be a bounded sequence in $W^{1,1}(\Omega)$. We will show the strong convergence of this sequence in three steps with the help of a family of mollifiers

$$u_k^{\epsilon}(x) = \int_{\Omega} j_{\epsilon}(y) u_k(x-y) dy.$$

First, we show that

$$u_k^\epsilon \to u_k \quad \text{in } L^1(\Omega) \text{ as } \epsilon \to 0\,, \text{ uniformly in k.} \tag{1.46}$$

Then, for each fixed $\epsilon > 0$, we prove that

$$\text{there is a subsequence of } \{u_k^\epsilon\} \text{ which converges uniformly .} \tag{1.47}$$

Finally, corresponding to the above convergent sequence $\{u_k^\epsilon\}$, we extract diagonally a subsequence of $\{u_k\}$ which converges strongly in $L^1(\Omega)$.

Based on the Extension Theorem, we may assume, without loss of generality, that $\Omega = R^n$, the sequence of functions $\{u_k\}$ all have compact support in a bounded open set $G \subset R^n$, and

$$\|u_k\|_{W^{1,1}(G)} \leq C < \infty\,, \text{ for all } k = 1, 2, \cdots .$$

Since every $W^{1,1}$ function can be approached by a sequence of smooth functions, we may also assume that each u_k is smooth.

Step 1. From the property of mollifiers, we have

$$\begin{aligned} u_k^\epsilon(x) - u_k(x) &= \int_{B_1(0)} j_\epsilon(y)[u_k(x - \epsilon y) - u_k(x)]dy \\ &= \int_{B_1(0)} j_\epsilon(y) \int_0^1 \frac{d}{dt} u_k(x - \epsilon t y) dt\, dy \\ &= -\epsilon \int_{B_1(0)} j_\epsilon(y) \int_0^1 D u_k(x - \epsilon t y) dt\, y\, dy. \end{aligned}$$

Integrating with respect to x and changing the order of integration, we obtain

$$\begin{aligned} \|u_k^\epsilon - u_k\|_{L^1(G)} &\leq \epsilon \int_{B_1(0)} j(y) \int_0^1 \int_G |D u_k(x - \epsilon t y)| dx\, dt\, dy \\ &\leq \epsilon \int_G |D u_k(z)| dz \leq \epsilon \|u_k\|_{W^{1,1}(G)}. \end{aligned} \tag{1.48}$$

It follows that

$$\|u_k^\epsilon - u_k\|_{L^1(G)} \to 0\,, \text{ as } \epsilon \to 0\,, \text{ uniformly in } k. \tag{1.49}$$

Step 2. Now fix an $\epsilon > 0$, then for all $x \in R^n$ and for all $k = 1, 2, \cdots$, we have

$$\begin{aligned} |u_k^\epsilon(x)| &\leq \int_{B_\epsilon(x)} j_\epsilon(x - y) |u_k(y)| dy \\ &\leq \|j_\epsilon\|_{L^\infty(R^n)} \|u_k\|_{L^1(G)} \leq \frac{C}{\epsilon^n} < \infty. \end{aligned} \tag{1.50}$$

Similarly,

$$|Du_k^\epsilon(x)| \leq \frac{C}{\epsilon^{n+1}} < \infty. \tag{1.51}$$

(1.50) and (1.51) imply that, for each fixed $\epsilon > 0$, the sequence $\{u_k^\epsilon\}$ is uniformly bounded and equi-continuous. Therefore, by the Arzela-Ascoli Theorem (see the Appendix), it possesses a subsequence (still denoted by $\{u_k^\epsilon\}$) which converges uniformly on G, in particular

$$\lim_{j,i\to\infty} \|u_j^\epsilon - u_i^\epsilon\|_{L^1(G)} = 0. \tag{1.52}$$

Step 3. Now choose ϵ to be $1, 1/2, 1/3, \cdots, 1/k, \cdots$ successively, and denote the corresponding subsequence that satisfies (1.52) by

$$u_{k1}^{1/k}, u_{k2}^{1/k}, u_{k3}^{1/k} \cdots,$$

for $k = 1, 2, 3, \cdots$. Pick the diagonal subsequence from the above:

$$\{u_{ii}^{1/i}\} \subset \{u_k^\epsilon\}.$$

Then select the corresponding subsequence from $\{u_k\}$:

$$u_{11}, u_{22}, u_{33}, \cdots.$$

This is our desired subsequence, because

$$\|u_{ii}-u_{jj}\|_{L^1(G)} \leq \|u_{ii}-u_{ii}^{1/i}\|_{L^1(G)}+\|u_{ii}^{1/i}-u_{jj}^{1/j}\|_{L^1(G)}+\|u_{jj}^{1/j}-u_{jj}\|_{L^1(G)}\to 0\,, \text{ as } i,j\to\infty,$$

due to the fact that each of the norm on the right hand side $\to 0$.

This completes the proof of the Lemma and hence the Theorem. □

1.6 Other Basic Inequalities

1.6.1 Poincaré's Inequality

For functions that vanish on the boundary, we have

Theorem 1.6.1 *(Poincaré's Inequality I).*

Assume Ω is bounded. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$. Then

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}. \tag{1.53}$$

Remark 1.6.1 *i) Now based on this inequality, one can take an equivalent norm of* $W_0^{1,p}(\Omega)$ *as* $\|u\|_{W_0^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)}$.

ii) Just for this theorem, one can easily prove it by using the Sobolev inequality $\|u\|_{L^p} \leq \|\bigtriangledown u\|_{L^q}$ *with* $p = \frac{nq}{n-q}$ *and the Hölder inequality. However, the one we present in the following is a unified proof that works for both this and the next theorem.*

Proof. For convenience, we abbreviate $\|u\|_{L^p(\Omega)}$ as $\|u\|_p$. Suppose inequality (1.53) does not hold, then there exists a sequence $\{u_k\} \subset W_0^{1,p}(\Omega)$, such that

$$\|Du_k\|_p = 1\,, \text{ while } \|u_k\|_p \to \infty\,, \text{ as } k \to \infty.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we may assume that $\{u_k\} \subset C_0^\infty(\Omega)$.

Let $v_k = \dfrac{u_k}{\|u_k\|_p}$. Then

$$\|v_k\|_p = 1\,, \text{ and } \|Dv_k\|_p \to 0\,.$$

Consequently, $\{v_k\}$ is bounded in $W^{1,p}(\Omega)$ and hence possesses a subsequence (still denoted by $\{v_k\}$) that converges weakly to some $v_o \in W^{1,p}(\Omega)$. From the compact embedding results in the previous section, $\{v_k\}$ converges strongly in $L^p(\Omega)$ to v_o, and therefore

$$\|v_o\|_p = 1\,. \tag{1.54}$$

On the other hand, for each $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega v\phi_{x_i} dx = \lim_{k\to\infty} \int_\Omega v_k \phi_{x_i} dx = -\lim_{k\to\infty} \int_\Omega v_{k,x_i} \phi dx = 0.$$

It follows that

$$Dv_o(x) = 0\,, a.e.$$

Thus v_o is a constant. Taking into account that $v_k \in C_0^\infty(\Omega)$, we must have $v_o \equiv 0$. This contradicts with (1.54) and therefore completes the proof of the Theorem. □

For functions in $W^{1,p}(\Omega)$, which may not be zero on the the boundary, we have another version of Poincaré's inequality.

Theorem 1.6.2 *(Poincaré Inequality II).*

Let Ω *be a bounded, connected, and open subset in* R^n *with* C^1 *boundary. Let* $\bar{u}$ *be the average of* u *on* Ω. *Assume* $1 \leq p \leq \infty$. *Then there exists a constant* $C = C(n, p, \Omega)$, *such that*

$$\|u - \bar{u}\|_p \leq C\|Du\|_p\,, \forall\, u \in W^{1,p}(\Omega). \tag{1.55}$$

The proof of this Theorem is similar to the previous one. Instead of letting $v_k = \frac{u_k}{\|u_k\|_p}$, we choose

$$v_k = \frac{u_k - \bar{u}_k}{\|u_k - \bar{u}_k\|_p}.$$

Remark 1.6.2 *The connect-ness of Ω is essential in this version of the inequality. A simple counter example is when $n = 1$, $\Omega = [0,1] \cup [2,3]$, and*

$$u(x) = \begin{cases} -1 & \text{for } x \in [0,1] \\ 1 & \text{for } x \in [2,3]\,. \end{cases}$$

1.6.2 The Classical Hardy-Littlewood-Sobolev Inequality

Theorem 1.6.3 *(Hardy-Littlewood-Sobolev Inequality).*
Let $0 < \lambda < n$ and $s, r > 1$ such that

$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$

Assume that $f \in L^r(R^n)$ and $g \in L^s(R^n)$. Then

$$\int_{R^n}\int_{R^n} f(x)|x-y|^{-\lambda} g(y) dx dy \leq C(n,s,\lambda)||f||_r ||g||_s \tag{1.56}$$

where

$$C(n,s,\lambda) = \frac{n|B^n|^{\lambda/n}}{(n-\lambda)rs}\left(\left(\frac{\lambda/n}{1-1/r}\right)^{\lambda/n} + \left(\frac{\lambda/n}{1-1/s}\right)^{\lambda/n}\right)$$

with $|B^n|$ being the volume of the unit ball in R^n, and where

$$\|f\|_r := \|f\|_{L^r(R^n)}.$$

Proof. Without loss of generality, we may assume that both f and g are non-negative and $\|f\|_r = 1 = \|g\|_s$.

Let

$$\chi_G(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$$

be the characteristic function of the set G. Then one can see obviously that

$$f(x) = \int_0^\infty \chi_{\{f>a\}}(x)\, da \tag{1.57}$$

$$g(x) = \int_0^\infty \chi_{\{g>b\}}(x)\, db \tag{1.58}$$

$$|x|^{-\lambda} = \lambda \int_0^\infty c^{-\lambda-1} \chi_{\{|x|<c\}}(x)\, dc \tag{1.59}$$

To see the last identity, one may first write

$$|x|^{-\lambda} = \int_0^\infty \chi_{\{|x|^{-\lambda} > \tilde{c}\}}(x)\, d\tilde{c},$$

and then let $\tilde{c} = c^{-\lambda}$.

Substituting (1.57), (1.58), and (1.59) into the left hand side of (1.56), we have

$$\begin{aligned} I &:= \int_{R^n}\int_{R^n} f(x)|x-y|^{-\lambda} g(y)\, dx\, dy \\ &= \lambda \int_0^\infty \int_0^\infty \int_0^\infty \int_{R^n}\int_{R^n} c^{-\lambda-1}\chi_{\{f>a\}}(x)\chi_{\{|x|<c\}}(x-y)\chi_{\{g>b\}}(y)\, dx\, dy\, dc\, db\, da \\ &= \lambda \int_0^\infty \int_0^\infty \int_0^\infty c^{-\lambda-1} I(a,b,c)\, dc\, db\, da. \end{aligned} \tag{1.60}$$

Let

$$u(c) = |B^n| c^n,$$

the volume of the ball of radius c, and let

$$v(a) = \int_{R^n} \chi_{\{f>a\}}(x)\, dx\,, \quad w(b) = \int_{R^n} \chi_{\{g>b\}}(y)\, dy,$$

the measure of the sets $\{x \mid f(x) > a\}$ and $\{y \mid g(y) > b\}$, respectively. Then we can express the norms as

$$\|f\|_r^r = r\int_0^\infty a^{r-1} v(a)\, da = 1 \text{ and } \|g\|_s^s = s\int_0^\infty b^{s-1} w(b)\, db = 1. \tag{1.61}$$

It is easy to see that

$$\begin{aligned} I(a,b,c) &\leq \int_{R^n}\int_{R^n} \chi_{\{|x|<c\}}(x-y)\chi_{\{g>b\}}(y)\, dx\, dy \\ &\leq \int_{R^n} u(c)\chi_{\{g>b\}}(y)\, dy = u(c)w(b) \end{aligned}$$

Similarly, one can show that $I(a,b,c)$ is bounded above by other pairs and arrive at

$$I(a,b,c) \leq \min\{u(c)w(b), u(c)v(a), v(a)w(b)\}. \tag{1.62}$$

We integrate with respect to c first. By (1.62), we have

$$\begin{aligned} &\int_0^\infty c^{-\lambda-1} I(a,b,c)\, dc \\ &\quad \leq \int_{u(c)\leq v(a)} c^{-\lambda-1} w(b)u(c)\, dc + \int_{u(c)>v(a)} c^{-\lambda-1} w(b)v(a)\, dc \end{aligned}$$

$$= w(b)|B^n| \int_0^{(v(a)/|B^n|)^{1/n}} c^{-\lambda-1+n}\, dc + w(b)v(a) \int_{(v(a)/|B^n|)^{1/n}} c^{-\lambda-1}\, dc$$
$$= \frac{|B^n|^{\lambda/n}}{n-\lambda} w(b)v(a)^{1-\lambda/n} + \frac{|B^n|^{\lambda/n}}{\lambda} w(b)v(a)^{1-\lambda/n}$$
$$= \frac{|B^n|^{\lambda/n}}{\lambda(n-\lambda)} w(b)v(a)^{1-\lambda/n} \tag{1.63}$$

Exchanging $v(a)$ with $w(b)$ in the second line of (1.63), we also obtain

$$\int_0^\infty c^{-\lambda-1} I(a,b,c)\, dc$$
$$\leq \int_{u(c)\leq w(b)} c^{-\lambda-1} v(a)u(c)\, dc + \int_{u(c)>w(b))} c^{-\lambda-1} w(b)v(a)\, dc$$
$$\leq \frac{|B^n|^{\lambda/n}}{\lambda(n-\lambda)} v(a)w(b)^{1-\lambda/n} \tag{1.64}$$

In view of (1.61), we split the b-integral into two parts, one from 0 to $a^{r/s}$ and the other from $a^{r/s}$ to ∞. By virtue of (1.60), (1.63), and (1.64), we derive

$$I \leq \frac{n}{n-\lambda} |B^n|^{\lambda/n}$$
$$\times \int_0^\infty v(a) \int_0^{a^{r/s}} w(b)^{1-\lambda/n}\, db\, da + \int_0^\infty v(a)^{1-\lambda/n} \int_{a^{r/s}}^\infty w(b)\, db\, da \tag{1.65}$$

To estimate the first integral in (1.65), we use Hölder inequality with $m = (s-1)(1-\lambda/n)$

$$\int_0^{a^{r/s}} w(b)^{1-\lambda/n} b^m b^{-m}\, db$$
$$\leq \left(\int_0^{a^{r/s}} w(b) b^{s-1}\, db \right)^{1-\lambda/n} \left(\int_0^{a^{r/s}} b^{-mn/\lambda}\, db \right)^{\lambda/n}$$
$$\leq \left(\int_0^{a^{r/s}} w(b) b^{s-1}\, db \right)^{1-\lambda/n} \cdot a^{r-1}, \tag{1.66}$$

since $mn/\lambda < 1$. It follows that the first integral in (1.65) is bounded above by

$$\left(\frac{\lambda}{n - s(n-\lambda)} \right)^{\lambda/n} \left(\int_0^\infty v(a) a^{r-1}\, da \right) \left(\int_0^\infty w(b) b^{s-1}\, db \right)^{1-\lambda/n}$$
$$= \frac{1}{rs} \left(\frac{\lambda/n}{1-1/r} \right)^{\lambda/n}. \tag{1.67}$$

To estimate the second integral in (1.65), we first rewrite it as

$$\int_0^\infty w(b) \int_0^{b^{s/r}} v(a)^{1-\lambda/n} \, da \, db,$$

then an analogous computation shows that it is bounded above by

$$\frac{1}{rs} \left(\frac{\lambda/n}{1 - 1/s} \right)^{\lambda/n}. \tag{1.68}$$

Now the desired Hardy-Littlewood-Sobolev inequality follows directly from (1.65), (1.67), and (1.68). □

Theorem 1.6.4 *(*An equivalent form of the Hardy-Littlewood-Sobolev inequality*)*
Let $g \in L^p(R^n)$ *for* $\frac{n}{n-\alpha} < p < \infty$*. Define*

$$Tg(x) = \int_{R^n} |x-y|^{\alpha-n} g(y) dy.$$

Then

$$\|Tg\|_p \leq C(n,p,\alpha) \|g\|_{\frac{np}{n+\alpha p}}. \tag{1.69}$$

Proof. By the classical Hardy-Littlewood-Sobolev inequality, we have

$$< f, Tg > = < Tf, g > \leq C(n,s,\alpha) \, \|f\|_r \, \|g\|_s \, ,$$

where $< \cdot, \cdot >$ is the L^2 inner product.

Consequently,

$$\|Tg\|_p = \sup_{\|f\|_r = 1} < f, Tg > \leq C(n,s,\alpha) \|g\|_s,$$

where

$$\begin{cases} \frac{1}{p} + \frac{1}{r} = 1 \\ \frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}. \end{cases}$$

Solving for s, we arrive at

$$s = \frac{np}{n + \alpha p}.$$

This completes the proof of the Theorem. □

Remark 1.6.3 *To see the relation between inequality (1.69) and the Sobolev inequality, let's rewrite it as*

$$\|Tg\|_{\frac{nq}{n-\alpha q}} \leq C \|g\|_q \tag{1.70}$$

with

$$1 < q := \frac{np}{n + \alpha p} < \frac{n}{\alpha}.$$

Let $u = Tg$. Then one can show that (see [CLO]),

$$(-\triangle)^{\frac{\alpha}{2}} u = g.$$

Now inequality (1.70) becomes the Sobolev one

$$\|u\|_{\frac{nq}{n-\alpha q}} \leq C \|(-\triangle)^{\frac{\alpha}{2}} u\|_q.$$

2

Existence of Weak Solutions

Many physical models come with natural variational structures where one can minimize a functional, usually an energy $E(u)$. Then the minimizers are weak solutions of the related partial differential equation

$$E'(u) = 0.$$

For example, for an open bounded domain $\Omega \subset R^n$, and for $f \in L^2(\Omega)$, the functional

$$E(u) = \frac{1}{2} \int_\Omega |\bigtriangledown u|^2 dx - \int_\Omega f\, u\, dx$$

possesses a minimizer among all $u \in H_0^1(\Omega)$ (try to show this by using the knowledge from the previous chapter). As we will see in this chapter, the minimizer is a weak solution of the Dirichlet boundary value problem

$$\begin{cases} -\triangle u = f(x)\,, & x \in \Omega \\ u(x) = 0\,, & x \in \partial\Omega. \end{cases}$$

In practice, it is often much easier to obtain a weak solution than a classical one. One of the seven millennium problem posed by the Clay Institute of Mathematical Sciences is to show that some suitable weak solutions to the Navie-Stoke equation with good initial values are in fact classical solutions for all time.

In this chapter, we consider the existence of weak solutions to second order elliptic equations, both linear and semi-linear.

For linear equations, we will use the Lax-Milgram Theorem to show the existence of weak solutions.

For semi-linear equations, we will introduce variational methods. We will consider the corresponding functionals in proper Hilbert spaces and seek weak solutions of the equations associated with the critical points of the functionals. We will use examples to show how to find a minimizer, a minimizer under constraint, and a mini-max critical point. The well-known *Mountain Pass Theorem* is introduced and applied to show the existence of such a mini-max.

To show that a weak solution possesses the desired differentiability so that it is actually a classical one is called the regularity method, which will be studied in Chapter 3.

2.1 Second Order Elliptic Operators

Let Ω be an open bounded set in R^n. Let $a_{ij}(x), b_i(x)$, and $c(x)$ be bounded functions on Ω with $a_{ij}(x) = a_{ji}(x)$. Consider the second order partial differential operator L either in the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u, \tag{2.1}$$

or in the non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u. \tag{2.2}$$

We say that L is uniformly elliptic if there exists a constant $\delta > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2$$

for almost every $x \in \Omega$ and for all $\xi \in R^n$.

In the simplest case when $a_{ij}(x) = \delta_{ij}$, $b_i(x) \equiv 0$, and $c(x) \equiv 0$, L reduces to $-\triangle$. And, as we will see later, that the solutions of the general second order elliptic equation $Lu = 0$ share many similar properties with the harmonic functions.

2.2 Weak Solutions

Let the operator L be in the divergence form as defined in (2.1). Consider the second order elliptic equation with Dirichlet boundary condition

$$\begin{cases} Lu = f \ , \ x \in \Omega \\ u = 0 \ , \quad x \in \partial\Omega \end{cases} \tag{2.3}$$

When $f = f(x)$, the equation is linear; and when $f = f(x,u)$, semi-linear.

Assume that, $f(x)$ is in $L^2(\Omega)$, or for a given solution u, $f(x,u)$ is in $L^2(\Omega)$.

Multiplying both sides of the equation by a test function $v \in C_0^\infty(\Omega)$, and integrating by parts on Ω, we have

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x) uv \right) dx = \int_\Omega f v dx. \tag{2.4}$$

There are no boundary terms because both u and v vanish on $\partial\Omega$. One can see that the integrals in (2.4) are well defined if u, v, and their first derivatives are square integrable. Write $H^1(\Omega) = W^{1,2}(\Omega)$, and let $H_0^1(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm of $H^1(\Omega)$:

$$\|u\| = \left[\int_\Omega (|Du|^2 + |u|^2) dx \right]^{1/2}.$$

Then (2.4) remains true for any $v \in H_0^1(\Omega)$. One can easily see that $H_0^1(\Omega)$ is also the most appropriate space for u. On the other hand, if $u \in H_0^1(\Omega)$ satisfies (2.4) and is second order differentiable, then through integration by parts, we have

$$\int_\Omega (Lu - f) v dx = 0 \ \ \forall v \in H_0^1(\Omega).$$

This implies that

$$Lu = f \ , \ \ \forall x \in \Omega.$$

And therefore u is a classical solution of (2.3).

The above observation naturally leads to

Definition 2.2.1 *The weak solution of problem (2.3) is a function* $u \in H_0^1(\Omega)$ *that satisfies (2.4) for all* $v \in H_0^1(\Omega)$.

Remark 2.2.1 *Actually, here the condition on* f *can be relaxed a little bit. By the Sobolev embedding*

$$H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega),$$

we see that v *is in* $L^{\frac{2n}{n-2}}(\Omega)$*, and hence by duality,* f *only need to be in* $L^{\frac{2n}{n+2}}(\Omega)$*. In the semi-linear case, if* $f(x,u) = u^p$ *for any* $p \leq \dfrac{n+2}{n-2}$ *or more generally*

$$f(x,u) \leq C_1 + C_2|u|^{\frac{n+2}{n-2}};$$

then for any $u \in H^1(\Omega)$*,* $f(x,u)$ *is in* $L^{\frac{2n}{n+2}}(\Omega)$*.*

2.3 Methods of Linear Functional Analysis

2.3.1 Linear Equations

For the Dirichelet problem with linear equation

$$\begin{cases} Lu = f(x) \ , \ x \in \Omega \\ u = 0 \ , \qquad x \in \partial\Omega, \end{cases} \tag{2.5}$$

to find its weak solutions, we can apply representation type theorems in Linear Functional Analysis, such as *Riesz Representation Theorem* or *Lax-Milgram Theorem*. To this end, we introduce the bilinear form

$$B[u,v] = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x) uv \right) dx$$

defined on $H_0^1(\Omega)$, which is the left hand side of (2.4) in the definition of the weak solutions. While the right hand side of (2.4) may be regarded as a linear functional on $H_0^1(\Omega)$;

$$< f, v >:= \int_\Omega f v dx.$$

Now, to find a weak solution of (2.5), it is equivalent to show that there exists a function $u \in H_0^1(\Omega)$, such that

$$B[u,v] =< f, v >, \ \ \forall \, v \in H_0^1(\Omega). \tag{2.6}$$

This can be realized by representation type theorems in Functional Analysis.

2.3.2 Some Basic Principles in Functional Analysis

Here we list briefly some basic definitions and principles of functional analysis, which will be used to obtain the existence of weak solutions. For more details, please see [Sch1].

Definition 2.3.1 *A Banach space* X *is a complete, normed linear space with norm* $\|\cdot\|$ *that satisfies*

(i) $\|u+v\| \leq \|u\| + \|v\|, \ \forall u, v \in X,$
(ii) $\|\lambda u\| = |\lambda| \|u\|, \ \forall u, v \in X, \lambda \in R,$
(iii) $\|u\| = 0$ *if and only if* $u = 0.$

The examples of Banach spaces are:
(i) $L^p(\Omega)$ with norm $\|u\|_{L^p(\Omega)}$,
(ii) Sobolev spaces $W^{k,p}(\Omega)$ with norm $\|u\|_{W^{k,p}(\Omega)}$, and
(iii) Hölder spaces $C^{0,\gamma}(\Omega)$ with norm $\|u\|_{C^{0,\gamma}(\Omega)}$.

Definition 2.3.2 *A Hilbert space H is a Banach space endowed with an inner product $(\cdot,\cdot)$ which generates the norm*

$$\|u\| := (u,u)^{1/2}$$

with the following properties
(i) $(u,v) = (v,u)$, $\forall u, v \in H$,
(ii) the mapping $u \mapsto (u,v)$ is linear for each $v \in H$,
(iii) $(u,u) \geq 0$, $\forall u \in H$, and
(iv) $(u,u) = 0$ if and only if $u = 0$.

Examples of Hilbert spaces are
(i) $L^2(\Omega)$ with inner product

$$(u,v) = \int_\Omega u(x)v(x)dx,$$

and
(ii) Sobolev spaces $H^1(\Omega)$ or $H^1_0(\Omega)$ with inner product

$$(u,v) = \int_\Omega [u(x)v(x) + Du(x)\cdot Dv(x)]dx.$$

Definition 2.3.3 *(i) A mapping $A : X \to Y$ is a linear operator if*

$$A[au + bv] = aA[u] + bA[u] \quad \forall\, u, v \in X, \;\; a, b \in R^1.$$

(ii) A linear operator $A : X \to Y$ is bounded provided

$$\|A\| := \sup_{\|u\|_X \leq 1} \|Au\|_Y < \infty.$$

(iii) A bounded linear operator $f : X \to R^1$ is called a bounded linear functional on X. The collection of all bounded linear functionals on X, denoted by X^*, is called the dual space of X. We use

$$< f, u >:= f[u]$$

to denote the pairing of X^* and X.

Lemma 2.3.1 *(*Projection*).*
Let M be a closed subspace of H. Then for every $u \in H$, there is a $v \in M$, such that

$$(u - v, w) = 0, \quad \forall w \in M. \tag{2.7}$$

In other words,

$$H = M \oplus M^\perp := \{u \in H \mid u \perp M\}.$$

Here v is the projection of u on M. The readers may find the proof of the Lemma in the book [Sch] (Theorem 1-3) or as given in the following exercise.

Exercise 2.3.1 *i) Given any $u \in H$ and $u \not\in M$, there exist a $v_o \in M$ such that*

$$\|v_o - u\| = \inf_{v \in M} \|v - u\|.$$

Hint: Show that a minimizing sequence is a Cauchy sequence.

ii) Show that

$$(u - v_o, w) = 0 \,, \quad \forall w \in M.$$

Based on this Projection Lemma, one can prove

Theorem 2.3.1 *(Riesz Representation Theorem). Let H be a real Hilbert space with inner product $(\cdot, \cdot)$, and H^* be its dual space. Then H^* can be canonically identified with H, more precisely, for each $u^* \in H^*$, there exists a unique element $u \in H$, such that*

$$< u^*, v >= (u, v) \quad \forall v \in H.$$

The mapping $u^* \mapsto u$ is a linear isomorphism from H^* onto H.

This is a well-known theorem in functional analysis. The readers may see the book [Sch1] for its proof or do it as the following exercise.

Exercise 2.3.2 *Prove the Riesz Representation Theorem in 4 steps.*

Let T be a linear operator from H to R^1 and $T \not\equiv 0$. Show that

i) $K(T) := \{u \in H \mid Tu = 0\}$ is closed.

ii) $H = K(T) \oplus K(T)^{\perp}$.

iii) The dimension of $K(T)$ is 1.

iv) There exists v, such that $Tv = 1$. Then

$$Tv = (u, v) \,, \quad \forall u \in H.$$

From Riesz Representation Theorem, one can derive the following

Theorem 2.3.2 *(Lax-Milgram Theorem). Let H be a real Hilbert space with norm $\| \cdot \|$. Let*

$$B : H \times H \rightarrow R^1$$

be a bilinear mapping. Assume that there exist constants $M, m > 0$, such that

(i) $$|B[u, v]| \leq M\|u\|\|v\|, \; \forall u, v \in H$$

and

(ii) $$m\|u\|^2 \leq B[u, u], \; \forall u \in H.$$

Then for each bounded linear functional f on H, there exists a unique element $u \in H$, such that

$$B[u, v] =< f, v >, \; \forall v \in H.$$

Proof. 1. If $B[u, v]$ is symmetric, i.e. $B[u, v] = B[v, u]$, then the conditions (i) and (ii) ensure that $B[u, v]$ can be made as an inner product in H, and the conclusion of the theorem follows directly from the Reisz Representation Theorem.

2. In the case $B[u, v]$ may not be symmetric, we proceed as follows.

On one hand, by the Riesz Representation Theorem, for a given bounded linear functional f on H, there is an $\tilde{f} \in H$, such that

$$< f, v >= (\tilde{f}, v), \quad \forall v \in H. \tag{2.8}$$

On the other hand, for each fixed element $u \in H$, by condition (i), $B[u, \cdot]$ is also a bounded linear functional on H, and hence there exist a $\tilde{u} \in H$, such that

$$B[u, v] = (\tilde{u}, v), \quad \forall v \in H. \tag{2.9}$$

Denote this mapping from u to $\tilde{u}$ by A, i.e.

$$A : H \rightarrow H, \quad Au = \tilde{u}. \tag{2.10}$$

From (2.8), (2.9), and (2.10), It suffice to show that for each $\tilde{f} \in H$, there exists a unique element $u \in H$, such that

$$Au = \tilde{f}.$$

We carry this out in two parts. In part (a), we show that A is a bounded linear operator, and in part (b), we prove that it is one-to-one and onto.

(a) For any $u_1, u_2 \in H$, any $a_1, a_2 \in R^1$, and each $v \in H$, we have

$$\begin{aligned} (A(a_1 u_1 + a_2 u_2), v) &= B[(a_1 u_1 + a_2 u_2), v] \\ &= a_1 B[u_1, v] + a_2 B[u_2, v] \\ &= a_1 (Au_1, v) + a_2 (Au_2, v) \\ &= (a_1 Au_1 + a_2 Au_2, v) \end{aligned}$$

Hence

$$A(a_1 u_1 + a_2 u_2) = a_1 Au_1 + a_2 Au_2.$$

A is linear.

Moreover, by condition (i), for any $u \in H$,

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq M \|u\| \|Au\|,$$

and consequently

$$\|Au\| \leq M \|u\| \quad \forall u \in H. \tag{2.11}$$

This verifies that A is bounded.

(b) To see that A is one-to-one, we apply condition (ii):

$$m \|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\| \|u\|,$$

and it follows that

$$m\|u\| \leq \|Au\|. \tag{2.12}$$

Now if $Au = Av$, then from (2.12),

$$m\|u - v\| \leq \|Au - Av\|.$$

This implies $u = v$. Hence A is one-to-one.

From (2.12), one can also see that $R(A)$, the range of A in H is closed. In fact, assume that $v_k \in R(A)$ and $v_k \to v$ in H. Then for each v_k, there is a u_k, such that $Au_k = v_k$. Inequality (2.12) implies that

$$\|u_i - u_j\| \leq C\|v_i - v_j\|,$$

i.e. u_k is a Cauchy sequence in H, and hence $u_k \to u$ for some $u \in H$. Now, by (2.11), $Au_k \to Au$ and $v = Au \in H$. Therefore, $R(A)$ is closed.

To show that $R(A) = H$, we need the *Projection Lemma 2.3.1*. For any $u \in H$, by the *Projection Lemma*, there exists a $v \in R(A)$, such that

$$0 = (u - v, A(u - v)) = B[(u - v), (u - v)] \geq m\|u - v\|^2.$$

Consequently, $u = v \in R(A)$ and hence $H \subset R(A)$. Therefore $R(A) = H$.

This completes the proof of the Theorem. □

2.3.3 Existence of Weak Solutions

Now we explore that in what situations, our second order elliptic operator L would satisfy the conditions (i) and (ii) requested by the Lax-Milgram Theorem.

Let $H = H_0^1(\Omega)$ be our Hilbert space with inner product

$$(u, v) := \int_\Omega (DuDv + uv)dx.$$

By Poincaré inequality, we can use the equivalent inner product

$$(u, v)_H := \int_\Omega DuDvdx$$

and the corresponding norm

$$\|u\|_H := \sqrt{\int_\Omega |Du|^2 dx}.$$

For any $v \in H$, by the *Sobolev Embedding*, v is in $L^{\frac{2n}{n-2}}(\Omega)$. And by virtue of the Hölder inequality,

$$< f, v >:= \int_\Omega f(x)v(x)dx \leq \left(\int_\Omega |f(x)|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{2n}} \left(\int_\Omega |v(x)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2n}},$$

hence we only need to assume that $f \in L^{\frac{2n}{n+2}}(\Omega)$.

In the simplest case when $L = -\triangle$, we have

$$B[u, v] = \int_\Omega Du \cdot Dv dx.$$

Obviously, condition (i) is met:

$$|B[u, v]| \leq \|u\|_H \|v\|_H,$$

Condition (ii) is also satisfied:

$$B[u, u] = \|u\|_H^2.$$

Now applying the *Lax-Milgram Theorem*, we conclude that there exists a unique $u \in H$ such that

$$B[u, v] =< f, v >, \quad \forall v \in H.$$

We have actually proved

Theorem 2.3.3 *For every $f(x)$ in $L^{\frac{2n}{n+2}}(\Omega)$, the Dirichlet problem*

$$\begin{cases} -\triangle u = f(x), \; x \in \Omega, \\ u(x) = 0, \qquad x \in \partial\Omega \end{cases}$$

has a unique weak solution u in $H_0^1(\Omega)$.

In general,

$$B[u, v] = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x) uv \right) dx.$$

Under the assumption that

$$\|a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)} \leq C,$$

and by Hölder inequality, one can easily verify that

$$|B[u, v]| \leq 3C \|u\|_H \|v\|_H.$$

Hence condition (i) is always satisfied. However, condition (ii) may not automatically hold. It depends on the sign of $c(x)$, and the L^∞ norm of $b_i(x)$ and $c(x)$.

First from the uniform ellipticity condition, we have

$$\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} dx \geq \delta \int_{\Omega} |Du|^2 dx = \delta \|u\|_H^2, \tag{2.13}$$

for some $\delta > 0$.

From here one can see that if $b_i(x) \equiv 0$ and $c(x) \geq 0$, then condition (ii) is met. Actually, the condition on $c(x)$ can be relaxed a little bit, for instance, if

$$c(x) \geq -\delta/2 \tag{2.14}$$

with δ as in (2.13), then condition (ii) holds. Hence we have

Theorem 2.3.4 *Assume that $b_i(x) \equiv 0$ and $c(x)$ satisfies (2.14). Then for every $f \in L^{\frac{2n}{n+2}}(\Omega)$, the problem*

$$\begin{cases} Lu = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

has a unique weak solution in $H_0^1(\Omega)$.

2.4 Variational Methods

2.4.1 Semi-linear Equations

To find the weak solutions of semi-linear equations

$$Lu = f(x, u)$$

the *Lax-Milgram Theorem* can no longer be applied. We will use variational methods to obtain the existence. Roughly speaking, we will associate the equation with the functional

$$J(u) = \frac{1}{2} \int_{\Omega} (Lu \cdot u) dx - \int_{\Omega} F(x, u) dx$$

where

$$F(x, u) = \int_0^u f(x, s) ds.$$

We show that the critical point of $J(u)$ is a weak solution of our problem. Then we will focus on how to seek critical points in various situations.

2.4.2 Calculus of Variations

For a continuously differentiable function g defined on R^n, a critical point is a point where Dg vanishes. The simplest sort of critical points are global or local maxima or minima. Others are saddle points. To seek weak solutions of partial differential equations, we generalize this concept to infinite dimensional spaces. To illustrate the idea, let us start from a simple example:

$$\begin{cases} -\triangle u = f(x) \ , \ x \in \Omega; \\ u(x) = 0 \ , \qquad x \in \partial\Omega. \end{cases} \tag{2.15}$$

To find weak solutions of the problem, we study the functional

$$J(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega f(x) u dx$$

in $H_0^1(\Omega)$.

Now suppose that there is some function u which happen to be a minimizer of $J(\cdot)$ in $H_0^1(\Omega)$, then we will demonstrate that u is actually a weak solution of our problem.

Let v be any function in $H_0^1(\Omega)$. Consider the real-value function

$$g(t) = J(u + tv), \quad t \in R.$$

Since u is a minimizer of $J(\cdot)$, the function $g(t)$ has a minimum at $t = 0$; and therefore, we must have

$$g'(0) = 0, \quad i.e. \ \frac{d}{dt} J(u + tv) \mid_{t=0} = 0.$$

Here, explicitly,

$$J(u + tv) = \frac{1}{2}\int_\Omega |D(u + tv)|^2 dx - \int_\Omega f(x)(u + tv) dx,$$

and

$$\frac{d}{dt} J(u + tv) = \int_\Omega D(u + tv) \cdot Dv dx - \int_\Omega f(x) v dx.$$

Consequently, $g'(0) = 0$ yields

$$\int_\Omega Du \cdot Dv dx - \int_\Omega f(x) v dx = 0, \ \ \forall v \in H_0^1(\Omega). \tag{2.16}$$

This implies that u is a weak solution of problem (2.15).

Furthermore, if u is also second order differentiable, then through integration by parts, we obtain

$$\int_\Omega \left[-\triangle u - f(x)\right] v dx = 0.$$

Since this is true for any $v \in H_0^1(\Omega)$, we conclude

$$-\triangle u - f(x) = 0.$$

Therefore, u is a classical solution of the problem.

From the above argument, the readers can see that, in order to be a weak solution of the problem, here u need not to be a minima; it can be a maxima, or a saddle point of the functional; or more generally, any point that satisfies

$$\frac{d}{dt} J(u+tv) \mid_{t=0} .$$

More precisely, we have

Definition 2.4.1 *Let J be a linear functional on a Banach space X.*

i) We say that J is Frechet differentiable at $u \in X$ if there exists a continuous linear map $L = L(u)$ from X to X^* satisfying:

For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, u)$, such that

$$|J(u+v) - J(u) - < L(u), v > | \leq \epsilon \|v\|_X \quad \text{whenever } \|v\|_X < \delta,$$

where

$$< L(u), v > := L(u)(v)$$

is the duality between X and X^*.

The mapping $L(u)$ is usually denoted by $J'(u)$.

ii) A critical point of J is a point at which $J'(u) = 0$, that is

$$< J'(u), v > = 0 \ \ \forall v \in X.$$

iii) We call $J'(u) = 0$ the Euler-Lagrange equation of the functional J.

Remark 2.4.1 *One can verify that, if J is Frechet differentiable at u, then*

$$< J'(u), v > = \lim_{t \longrightarrow 0} \frac{J(u+tv) - J(u)}{t} = \frac{d}{dt} J(u+tv) \mid_{t=0} .$$

2.4.3 Existence of Minimizers

In the previous subsection, we have seen that the critical points of the functional

$$J(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} f(x) u dx$$

are weak solutions of the Dirichlet problem (2.15). Then our next question is:

Does the functional J actually possess a critical point?

We will show that there exists a minimizer of J. To this end, we verify that J possesses the following three properties:

i) bounded-ness from below,
ii) coercivity, and
iii) weak lower semi-continuity.

i) Bounded-ness from Below.

First we prove that J is bounded from below in $H := H_0^1(\Omega)$ if $f \in L^2(\Omega)$. Again, for simplicity, we use the equivalent norm in H:

$$\|u\|_H = \sqrt{\int_\Omega |Du|^2 dx}.$$

By *Poincaré* and *Hölder* inequalities, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_H^2 - C\|u\|_H\|f\|_{L^2(\Omega)} \\ &= \frac{1}{2}\left(\|u\|_H - C\|f\|_{L^2(\Omega)}\right)^2 - \frac{C^2}{2}\|f\|^2_{L^2(\Omega)} \\ &\geq -\frac{C^2}{2}\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.17}$$

Therefore, the functional J is bounded from below.

ii) Coercivity. However, A functional that is bounded from below does not guarantee that it has a minimum. A simple counter example is

$$g(x) = \frac{1}{1+x^2}\ , \quad x \in R^1.$$

Obviously, the infimum of $g(x)$ is 1, but it can never be attained. If we take a minimizing sequence $\{x_k\}$ such that $g(x_k)\to 1$, then x_k goes away to infinity. This suggests that in order to prevent a minimizing sequence from 'leaking' to infinity, we need to have some sort of 'coercive' condition on our functional J. That is, if a sequence $\{u_k\}$ goes to infinity, i.e. if $\|u\|_H\to\infty$, then $J(u_k)$ must also become unbounded. Therefore a minimizing sequence would be retained in a bounded set. And this is indeed true for our functional J. Actually, from the second part of (2.17), one can see that

$$\text{if } \|u_k\|_H\to\infty, \quad \text{then } J(u_k)\to\infty.$$

This implies that any minimizing sequence must be bounded in H.

iii) Weak Lower Semi-continuity. If H is a finite dimensional space, then a bounded minimizing sequence $\{u_k\}$ would possesses a convergent subsequence, and the limit would be a minimizer of J. This is no longer true in infinite dimensional spaces. We therefore turn our attention to weak topology.

Definition 2.4.2 *Let X be a real Banach space. If a sequence $\{u_k\} \subset X$ satisfies*

$$< f, u_k > \to < f, u_o > \quad \textit{as } k\to\infty$$

for every bounded linear functional f on X; then we say that $\{u_k\}$ converges weakly to u_o in X, and write

$$u_k \rightharpoonup u_o.$$

Definition 2.4.3 *A Banach space is called reflexive if* $(X^*)^* = X$*, where* X^* *is the dual of* X*.*

Theorem 2.4.1 *(Weak Compactness.) Let* X *be a reflexive Banach space. Assume that the sequence* $\{u_k\}$ *is bounded in* X*. Then there exists a subsequence of* $\{u_k\}$*, which converges weakly in* X*.*

Now let's come back to our Hilbert space $H := H_0^1(\Omega)$. By the *Reisz Representation Theorem*, for every linear functional f on H, there is a unique $v \in H$, such that

$$< f, u >= (v, u), \quad \forall u \in H.$$

It follows that $H^* = H$, and therefore H is reflexive. By the *Coercivity* and *Weak Compactness Theorem*, now a minimizing sequence $\{u_k\}$ is bounded, and hence possesses a subsequence $\{u_{k_i}\}$ that converges weakly to an element u_o in H:

$$(u_{k_i}, v) \rightarrow (u_o, v), \quad \forall v \in H.$$

Naturally, we wish that the functional J we consider is continuous with respect to this weak convergence, that is

$$\lim_{i \rightarrow \infty} J(u_{k_i}) = J(u_o),$$

then u_o would be our desired minimizer. However, this is not the case with our functional, nor is it true for most other functionals of interest. Fortunately, we do not really need such a continuity, instead, we only need a weaker one:

$$J(u_o) \leq \liminf_{i \rightarrow \infty} J(u_{k_i}).$$

Since on the other hand, by the definition of a minimizing sequence, we obviously have

$$J(u_o) \geq \liminf_{i \rightarrow \infty} J(u_{k_i}),$$

and hence

$$J(u_o) = \liminf_{i \rightarrow \infty} J(u_{k_i}).$$

Therefore, u_o is a minimizer.

Definition 2.4.4 *We say that a functional* $J(\cdot)$ *is weakly lower semi-continuous on a Banach space* X*, if for every weakly convergent sequence*

$$u_k \rightharpoonup u_o\,, \quad in \;\; X,$$

we have

$$J(u_o) \leq \liminf_{k \rightarrow \infty} J(u_k).$$

Now we show that our functional J is weakly lower semi-continuous in H. Assume that $\{u_k\} \subset H$, and

$$u_k \rightharpoonup u_o \quad \text{in } H.$$

Since for each fixed $f \in L^{\frac{2n}{n+2}}(\Omega)$, $\int_\Omega f u dx$ is a linear functional on H, it follows immediately

$$\int_\Omega f u_k dx \to \int_\Omega f u_o dx, \quad \text{as } k \to \infty. \tag{2.18}$$

From the algebraic inequality

$$a^2 + b^2 \geq 2ab,$$

we have

$$\int_\Omega |Du_k|^2 dx \geq \int_\Omega |Du_o|^2 dx + 2 \int_\Omega Du_o \cdot (Du_k - Du_o) dx.$$

Here the second term on the right goes to zero due to the weak convergence $u_k \rightharpoonup u_o$. Therefore

$$\liminf_{k \to \infty} \int_\Omega |Du_k|^2 dx \geq \int_\Omega |Du_o|^2 dx. \tag{2.19}$$

Now (2.18) and (2.19) imply the weakly lower semi-continuity. So far, we have proved

Theorem 2.4.2 *Assume that Ω is a bounded domain with smooth boundary. Then for every $f \in L^{\frac{2n}{n+2}}(\Omega)$, the functional*

$$J(u) = \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega f u dx$$

possesses a minimum u_o in $H_0^1(\Omega)$, which is a weak solution of the boundary value problem

$$\begin{cases} -\triangle u = f(x)\,, & x \in \Omega \\ u(x) = 0\,, & x \in \partial\Omega. \end{cases}$$

2.4.4 Existence of Minimizers Under Constraints

Now we consider the semi-linear Dirichlet problem

$$\begin{cases} -\triangle u = |u|^{p-1} u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.20}$$

with $1 < p < \frac{n+2}{n-2}$.

Obviously, $u \equiv 0$ is a solution. We call this a trivial solution. Of course, what we are more interested is whether there exists non-trivial solutions. Let

$$J(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

It is easy to verify that

$$\frac{d}{dt}J(u+tv)\mid_{t=0} = \int_\Omega \left(Du \cdot Dv - |u|^{p-1}uv\right) dx.$$

Therefore, a critical point of the functional J in $H := H_0^1(\Omega)$ is a weak solution of (2.20)

However, this functional is not bounded from below. To see this, fixed an element $u \in H$ and consider

$$J(tu) = \frac{t^2}{2}\int_\Omega |Du|^2 dx - \frac{t^{p+1}}{p+1}\int_\Omega |u|^{p+1} dx.$$

Noticing $p+1 > 2$, we find that

$$J(tu) \rightarrow -\infty, \quad \text{as } t \rightarrow \infty.$$

Therefore, there is no minimizer of $J(u)$. For this reason, instead of J, we will consider another functional

$$I(u) := \frac{1}{2}\int_\Omega |Du|^2 dx,$$

in the constrain set

$$M := \{u \in H \mid G(u) := \int_\Omega |u|^{p+1} dx = 1\}.$$

We seek minimizers of I in M. Let $\{u_k\} \subset M$ be a minimizing sequence, i.e.

$$\lim_{k \longrightarrow \infty} I(u_k) = \inf_{u \in M} I(u) := m.$$

It follows that $\int_\Omega |Du_k|^2 dx$ is bounded, hence $\{u_k\}$ is bounded in H. By the *Weak Compactness Theorem*, there exist a subsequence (for simplicity, we still denote it by $\{u_k\}$), which converges weakly to u_o in H. The weak lower semi-continuity of the functional I yields

$$I(u_o) \leq \liminf_{k \longrightarrow \infty} I(u_k) = m. \tag{2.21}$$

On the other hand, since $p+1 < \dfrac{2n}{n-2}$, by the *Compact Sobolev Embedding Theorem*

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^{p+1}(\Omega),$$

we find that u_k converges strongly to u_o in $L^{p+1}(\Omega)$, and hence

$$\int_\Omega |u_o|^{p+1} dx = 1.$$

That is $u_o \in M$, and therefore

$$I(u_o) \geq m.$$

Together with (2.21), we obtain

$$I(u_o) = m.$$

Now we have shown the existence of a minimizer u_o of I in M. To link u_o to a weak solution of (2.20), we derive the corresponding Euler-Lagrange equation for this minimizer under the constraint.

Theorem 2.4.3 *(Lagrange Multiplier). let u be a minimizer of I in M, i.e.*

$$I(u) = \min_{v \in M} I(v).$$

Then there exists a real number λ such that

$$I'(u) = \lambda G'(u),$$

or

$$< I'(u), v >= \lambda < G'(u), v >, \quad \forall v \in H.$$

Before proving the theorem, let's first recall the *Lagrange Multiplier* for functions defined on R^n. Let $f(x)$ and $g(x)$ be smooth functions in R^n. It is well known that the critical points (in particular, the minima) x^o of $f(x)$ under the constraint $g(x) = c$ satisfy

$$Df(x^o) = \lambda Dg(x^o) \tag{2.22}$$

for some constant λ. Geometrically, $Df(x^o)$ is a vector in R^n. It can be decomposed as the sum of two perpendicular vectors:

$$Df(x^o) = Df(x^o) \mid_N + Df(x^o) \mid_T,$$

where the two terms on the right hand side are the projection of $Df(x^o)$ onto the normal and tangent spaces of the level set $g(x) = c$ at point x^o. By definition, x^o being a critical point of $f(x)$ on the level set $g(x) = c$ means that the tangential projection $Df(x^o) \mid_T = 0$. While the normal projection $Df(x^o) \mid_N$ is parallel to $Dg(x^o)$, and we therefore have (2.22). Heuristically, we may generalize this perception into infinite dimensional space H. At a fixed point $u \in H$, $I'(u)$ is a linear functional on H. By the *Reisz Representation*

Theorem, it can be identified as a point (or a vector) in H, denoted by $\widetilde{I'(u)}$, such that

$$< I'(u), v >= (\widetilde{I'(u)}, v), \quad \forall v \in H.$$

Similarly for $G'(u)$, we have $\widetilde{G'(u)} \in H$, and it is a normal vector of M at point u. At a minimum u of I on the hyper surface $\{w \in H \mid G(w) = 1\}$, $\widetilde{I'(u)}$ must be perpendicular to the tangent space at point u, and hence

$$\widetilde{I'(u)} = \lambda \widetilde{G'(u)}.$$

We now prove this rigorously.

The Proof of Theorem 2.4.3.

Recall that if u is the minimum of J in the whole space H, then, for any $v \in H$, we have

$$\frac{d}{dt} I(u + tv) \mid_{t=0} . \tag{2.23}$$

Now u is only the minimum on M, we can no longer use (2.23) because $u+tv$ can not be kept on M, we can not guarantee that

$$G(u + tv) := \int_\Omega |u + tv|^{p+1} dx = 1$$

for all small t. To remedy this, we consider

$$g(t, s) := G(u + tv + sw).$$

For each small t, we try to show that there is an $s = s(t)$, such that

$$G(u + tv + s(t)w) = 1.$$

Then we can calculate the variation of $J(u + tv + s(t)w)$ as $t \to 0$.

In fact,

$$\frac{\partial g}{\partial s}(0, 0) =< G'(u), w >= (p + 1) \int_\Omega |u|^{p-1} u w dx.$$

Since $u \in M$, $\int_\Omega |u|^{p+1} dx = 1$, there exists w (may just take w as u), such that the right hand side of the above is not zero. Hence, according to the *Implicit Function Theorem*, there exists a C^1 function $s : R^1 \to R^1$, such that

$$s(0) = 0, \quad \text{and } g(t, s(t)) = 1,$$

for sufficiently small t. Now $u+tv+s(t)w$ is on M, and hence at the minimum u of J on M, we must have

$$\frac{d}{dt} I(u + tv + s(t)w) \mid_{t=0} =< I'(u), v > + < I'(u), w > s'(0) = 0. \tag{2.24}$$

On the other hand, we have

$$0 = \frac{dg}{dt}(0,0) = \frac{\partial g}{\partial t}(0,0) + \frac{\partial g}{\partial s}(0,0)s'(0)$$
$$= < G'(u), v > + < G'(u), w > s'(0).$$

Consequently

$$s'(0) = -\frac{< G'(u), v >}{< G'(u), w >}.$$

Substitute this into (2.24), and denote

$$\lambda = \frac{< I'(u), w >}{< G'(u), w >},$$

we arrive at

$$< I'(u), v >= \lambda < G'(u), v >, \quad \forall v \in H.$$

This completes the proof of the Theorem. □

Now let's come back to the weak solution of problem (2.20). Our minimizer u_o of I under the constraint $G(u) = 1$ satisfies

$$< I'(u_o), v >= \lambda < G'(u_o), v >, \quad \forall v \in H,$$

that is

$$\int_\Omega Du_o \cdot Dv dx = \lambda \int_\Omega |u_o|^{p-1} u_o v dx, \quad \forall v \in H. \tag{2.25}$$

One can see that $\lambda > 0$. Let $\tilde{u} = a u_o$ with $\lambda / a^{p-1} = 1$. This is possible since $p > 1$. Then it is easy to verify that

$$\int_\Omega D\tilde{u} \cdot Dv dx = \int_\Omega |\tilde{u}|^{p-1} \tilde{u} v dx.$$

Now $\tilde{u}$ is the desired weak solution of (2.20) in H. □

Exercise 2.4.1 *Show that $\lambda > 0$ in (2.25).*

2.4.5 Mini-max Critical Points

Besides local or global minima and maxima, there are other types of critical points: saddle points or mini-max points. For a simple example, let's consider the function $f(x, y) = x^2 - y^2$ from R^2 to R^1. $(x, y) = (0, 0)$ is a critical point of f, i.e. $Df(0, 0) = 0$. However, it is neither a local maximum nor a local minimum. The graph of $z = f(x, y)$ in R^3 looks like a horse saddle. If we go on the graph along the direction of x-axis, (0,0) is the minimum, while along the direction of y-axis, (0,0) is the maximum. For this reason, we also call (0,0) a mini-max critical point of $f(x, y)$. The way to locate or to show the existence of such mini-max point is called a mini-max variational method. To illustrate the main idea, let's still take this simple example. Suppose we don't know whether $f(x, y)$ possesses a mini-max point, but we do detect a 'mountain range' on the graph near xz plane. To show the existence of a mini-max point, we pick up two points on xy plane, for instance, $P = (1, 1)$ and $Q = (-1, -1)$ on two sides of the 'mountain range'. Let $\gamma(s)$ be a continuous curve linking the two points P and Q with $\gamma(0) = P$ and $\gamma(1) = Q$. To go from $(P, f(P))$ to $(Q, f(Q))$ on the graph, one needs first to climb over the 'mountain range' near xz plane, then to decent to the point $(Q, f(Q))$. Hence on this path, there is a highest point. Then we take the infimum among the highest points on all the possible paths linking P and Q. More precisely, we define

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} f(\gamma(s)),$$

where

$$\Gamma = \{\gamma \mid \gamma = \gamma(s) \text{ is continuous on } [0,1], \text{ and } \gamma(0) = P, \gamma(1) = Q\}$$

We then try to show that there exists a point P_o at which f attains this mini-max value c, and more importantly, $Df(P_o) = 0$.

For a functional J defined on an infinite dimensional space X, we will apply a similar idea to seek its mini-max critical points. Unlike finite dimensional space, in an infinite dimensional space, a bounded sequence may not converge. Hence we require the functional J to satisfy some compactness condition, which is the Palais-Smale condition.

Definition 2.4.5 *We say that the functional J satisfies the Palais-Smale condition (in short, (PS)), if any sequence $\{u_k\} \in X$ for which $J(u_k)$ is bounded and $J'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.*

Under this compactness condition, Ambrosetti and Rabinowitz [AR] establish the following well-known theorem on the existence of a mini-max critical point.

Theorem 2.4.4 *(Mountain Pass Theorem). Let X be a real Banach space and $J \in C^1(X, R^1)$. Suppose J satisfies (PS), $J(0) = 0$,*

(J_1) there exist constants $\rho, \alpha > 0$ such that $J \mid_{\partial B_\rho(0)} \geq \alpha$, and

(J_2) there is an $e \in X \setminus \overline{B_\rho(0)}$, such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq \alpha$ which can be characterized as

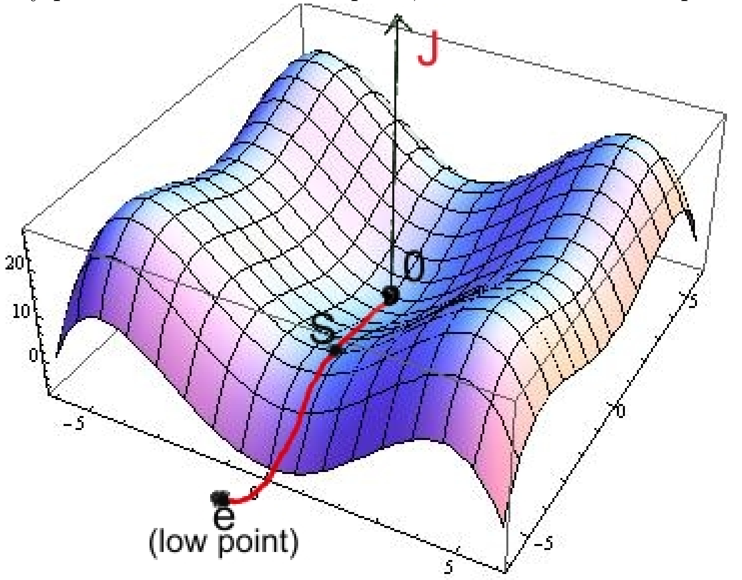
$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[0,1]} J(u)$$

where

$$\Gamma = \{\gamma \in C([0, 1], R^1) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Here, a critical value c means a value of the functional which is attained by a critical point u, i.e. $J'(u) = 0$ and $J(u) = c$.

Heuristically, the Theorem says if the point 0 is located in a valley surrounded by a mountain range, and if at the other side of the range, there is another low point e; then there must be a mountain pass from 0 to e which contains a mini-max critical point of J. In the figure below, on the graph of J, the path connecting the two low points $(0, J(0))$ and $(e, J(e))$ contains a highest point S, which is the lowest as compared to the highest points on the nearby paths. This S is a saddle point, the mini-max critical point of J.



The proof of the Theorem is mainly based on a deformation theorem which exploits the change of topology of the level sets of J when the value of J goes through a critical value. To illustrate this, let's come back to the simple example $f(x, y) = x^2 - y^2$. Denote the level set of f by

$$f_c := \{(x, y) \in R^2 \mid f(x, y) \leq c\}.$$

One can see that 0 is a critical value and for any $a > 0$, the level sets f_a and f_{-a} has different topology. f_a is connected while f_{-a} is not. If there is no critical value between the two level sets, say f_a and f_b with $0 < a < b$, then one can continuously deform the set f_b into f_a. One convenient way is let the points in the set f_b 'flow' along the direction of $-Df$ into f_a; correspondingly, the points on the graph 'flow downhills' into a lower level. More precisely and generally, we have

Theorem 2.4.5 *(Deformation Theorem). Let X be a real Banach space. Assume that $J \in C^1(X, R^1)$ and satisfies the (PS). Suppose that c is not a critical value of J.*

Then for each sufficiently small $\epsilon > 0$*, there exists a constant* $0 < \delta < \epsilon$ *and a continuous mapping* $\eta(t, u)$ *from* $[0, 1] \times X$ *to* X*, such that*

(i) $\eta(0, u) = u, \;\; \forall u \in X$,

(ii) $\eta(1, u) = u, \;\; \forall u \not\in J^{-1}[c - \epsilon, c + \epsilon]$,

(iii) $J(\eta(t, u) \leq J(u), \;\; \forall u \in X, \; t \in [0, 1]$,

(iv) $\eta(1, J_{c+\delta}) \subset J_{c-\delta}$.

Roughly speaking, the Theorem says if c is not a critical value of J, then one can continuously deform a higher level set $J_{c+\delta}$ into a lower level $J_{c-\delta}$ by 'flowing downhills'. Condition (ii) is to ensure that after the deformation, the two points 0 and e as in the *Mountain Pass theorem* still remain fixed.

Proof of the Deformation Theorem.

The main idea is to solve an ODE initial value problem (with respect to t) for each $u \in X$, roughly look like the following:

$$\begin{cases} \frac{d\eta}{dt}(t, u) = -J'(\eta(t, u)), \\ \eta(0, u) = u. \end{cases} \tag{2.26}$$

That is, to let the 'flow' go along the direction of $-J'(\eta(t, u))$, so that the value of $J(\eta(t, u)$ would decrease as t increases. However, we need to modify the right hand side of the equation a little bit, so that the flow will satisfy other desired conditions.

Step 1 We first choose a small $\epsilon > 0$, so that the flow would not go too slow in $J_{c+\epsilon} \setminus J_{c-\epsilon}$. This is actually guaranteed by the (PS). We claim that there exist constants $0 < a, \epsilon < 1$, such that

$$\|J'(u)\| \geq a, \;\; \forall u \in J_{c+\epsilon} \setminus J_{c-\epsilon}. \tag{2.27}$$

Otherwise, there would exist sequences $a_k \to 0$, $\epsilon_k \to 0$ and elements $u_k \in J_{c+\epsilon_k} \setminus J_{c-\epsilon_k}$ with $\|J'(u_k)\| \leq a_k$. Then by virtue of the (PS) condition, there is a subsequence $\{u_{k_i}\}$ that converges to an element $u_o \in X$. Since $J \in C^1(X, R^1)$, we must have

$$J(u_o) = c \quad \text{and } J'(u_o) = 0.$$

This is a contradiction with the assumption that c is not a critical value of J. Therefore (2.27) holds.

Step 2. To satisfy condition (ii), i.e. to make the flow stay still in the set

$$M := \{u \in X \mid J(u) \leq c - \epsilon \text{ or } J(u) \geq c + \epsilon\},$$

we could multiply the right hand side of equation (2.26) by $dist(\eta, M)$, but this would make the flow go too slow near M. To modify further, we choose $0 < \delta < \epsilon$, such that

$$\delta < \frac{a^2}{2}, \tag{2.28}$$

and let

$$N := \{u \in X \mid c - \delta \leq J(u) \leq c + \delta\}.$$

Now for $u \in X$, define

$$g(u) := \frac{dist(u, M)}{dist(u, M) + dist(u, N)},$$

and let

$$h(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ \frac{1}{s} & \text{if } s > 1. \end{cases}$$

We finally modify the $-J'(\eta)$ as

$$F(\eta) := -g(\eta)h(\|J'(\eta)\|)J'(\eta).$$

The introduction of the factor $h(\cdot)$ is to ensure the bounded-ness of $F(\cdot)$.

Step 3. Now for each fixed $u \in X$, we consider the modified ODE problem

$$\begin{cases} \frac{d\eta}{dt}(t, u) = F(\eta(t, u)) \\ \eta(0, u) = u. \end{cases}$$

One can verify that F is bounded and Lipschitz continuous on bounded sets, and therefore by the well-known ODE theory, there exists a unique solution $\eta(t, u)$ for all time $t \geq 0$.

Condition (i) is satisfied automatically. From the definition of g

$$g(u) = 0 \ \ \forall u \in M;$$

hence condition (ii) is also satisfied.

To verify (iii) and (iv), we compute

$$\begin{aligned} \frac{d}{dt}J(\eta(t, u)) &= < J'(\eta(t, u)), \frac{d\eta}{dt}(t, u) > \\ &= < J'(\eta(t, u)), F(\eta(t, u)) > \\ &= -g(\eta(t, u))h(\|J'(\eta(t, u))\|)\|J'(\eta(t, u))\|^2. \end{aligned} \tag{2.29}$$

In the above equality, obviously, the right hand side is non-positive, and hence $J(\eta(t, u))$ is non-increasing. This verifies (iii).

To see (iv), it suffice to show that for each u in $N = J_{c+\delta} \setminus J_{c-\delta}$, we have

$$\eta(1, u) \in J_{c-\delta}.$$

Notice that for $\eta \in N$, $g(\eta) \equiv 1$, and (2.29) becomes

$$\frac{d}{dt}J(\eta(t, u)) = -h(\|J'(\eta(t, u))\|)\|J'(\eta(t, u)\|^2.$$

By the definition of h and (2.27), we have,

$$\frac{d}{dt}J(\eta(t,u)) = \begin{cases} -\|J'(\eta(t,u))\|^2 \leq -a^2 & \text{if } \|J'(\eta(t,u))\| \leq 1 \\ -\|J'(\eta(t,u))\| \leq -a & \text{if } \|J'(\eta(t,u))\| > 1. \end{cases}$$

In any case, we derive

$$J(\eta(1,u)) \leq J(\eta(0,u)) - a^2 \leq c + \delta - a^2 \leq c - \delta.$$

This establishes (iv) and therefore completes the proof of the Theorem. □

With this *Deformation Theorem*, the proof of the *Mountain Pass Theorem* becomes very simple.

The Proof of the Mountain Pass Theorem.

First, due to the definition, we have $c < \infty$. To see this, we take a particular path $\gamma(t) = te$ and consider the function

$$g(t) = J(\gamma(t)).$$

It is continuous on the closed interval [0,1], and hence bounded from above. Therefore

$$c \leq \max_{u \in \gamma([0,1])} J(u) = \max_{t \in [0,1]} g(t) < \infty.$$

On the other hand, for any $\gamma \in \Gamma$

$$\gamma([0,1]) \cap \partial B_\rho(0) \neq \emptyset.$$

Hence by condition (J_1),

$$\max_{u \in \gamma([0,1])} J(u) \geq \inf_{v \in \partial B_\rho(0)} J(v) \geq \alpha.$$

And it follows that $c \geq \alpha$.

Suppose the number c so defined is not a critical value. We will use the *Deformation Theorem* to continuously deform a path $\gamma \in \Gamma$ down to the level set $J_{c-\delta}$ to derive a contradiction. More precisely, choose the ϵ in the *Deformation Theorem* to be $\frac{\alpha}{2}$, and $0 < \delta < \epsilon$. From the definition of c, we can pick a path $\gamma \in \Gamma$, such that

$$\max_{u \in \gamma([0,1])} J(u) \leq c + \delta,$$

i.e.

$$\gamma([0,1]) \subset J_{c+\delta}.$$

Now by the *Deformation Theorem*, this path can be continuously deformed down to the level set $J_{c-\delta}$, that is

$$\eta(1, \gamma([0,1]) \subset J_{c-\delta},$$

or in order words,

$$\max_{u\in\eta(1,\gamma([0,1]))} J(u) \leq c - \delta. \tag{2.30}$$

On the other hand, since 0 and e are in $J_{c-\epsilon}$, by (ii) of the *Deformation Theorem*,

$$\eta(1,0) = 0 \quad \text{and } \eta(1,e) = e.$$

This means that $\eta(1,\gamma([0,1]))$ is also a path in Γ, and therefore we must have

$$\max_{u\in\eta(1,\gamma([0,1]))} J(u) \geq c.$$

This contradicts with (2.30) and hence completes the proof of the Theorem.
□

2.4.6 Existence of a Mini-max via the Mountain Pass Theorem

Now we apply the *Mountain Pass Theorem* to seek weak solutions of the semi-linear elliptic problem

$$\begin{cases} -\triangle u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.31}$$

Although the method we illustrate here could be applied to a more general second order uniformly elliptic equations, we prefer this simple model that better illustrates the main ideas.

We assume that $f(x,u)$ satisfies

$(f_1) f(x,s) \in C(\bar{\Omega} \times R^1, R^1)$,

(f_2) there exists constants $c_1, c_2 \geq 0$, such that

$$|f(x,s)| \leq c_1 + c_2|s|^p,$$

with $0 \leq p < \frac{n+2}{n-2}$ if $n > 2$. If $n = 1$, (f_2) can be dropped; if $n = 2$, it suffice that

$$|f(x,s)| \leq c_1 e^{\phi(s)},$$

with $\phi(s)/s^2 \to 0$ as $|s| \to \infty$.

$(f_3) f(x,s) = o(|s|)$ as $s \to 0$, and

(f_4) there are constants $\mu > 2$ and $r \geq 0$ such that for $|s| \geq r$,

$$0 < \mu F(x,s) \leq s f(x,s),$$

where

$$F(x,s) = \int_0^s f(x,t)dt.$$

Remark 2.4.2 *A simple example of such function $f(x,u)$ is $|u|^{p-1}u$.*

Theorem 2.4.6 *(Rabinowitz [Ra]) If f satisfies condition $(f_1) - (f_4)$, then problem (2.31) possesses a nontrivial weak solution.*

To find weak solutions of the problem, we seek mini-max critical points of the functional

$$J(u) = \frac{1}{2}\int_{\Omega} |Du|^2 dx - \int_{\Omega} F(x,u)dx$$

on the Hilbert space $H := H_0^1(\Omega)$. We verify that J satisfies the conditions in the *Mountain Pass Theorem*.

We first need to show that $J \in C^1(H, R^1)$. Since we have verified that the first term

$$\int_{\Omega} |Du|^2 dx$$

is continuously differentiable, now we only need to check the second term

$$I(u) := \int_{\Omega} F(x,u)dx.$$

Proposition 2.4.1 *(Rabinowitz [Ra]) Let $\Omega \subset R^n$ be a bounded domain and let g satisfy*

$(g_1) g \in C(\bar{\Omega} \times R^1, R^1)$, and

(g_2) there are constants $r, s \geq 1$ and $a_1, a_2 \geq 0$ such that

$$|g(x,\xi)| \leq a_1 + a_2|\xi|^{r/s}$$

for all $x \in \bar{\Omega}$, $\xi \in R^1$.

Then the map $u(x) \to g(x, u(x))$ belongs to $C(L^r(\Omega), L^s(\Omega))$.

Proof. By (g_2),

$$\int_{\Omega} |g(x,u(x))|^s dx \leq \int_{\Omega} (a_1 + a_2|u|^{r/s})^s dx$$
$$\leq a_3 \int_{\Omega} (1 + |u|^r)dx.$$

It follows that if $u \in L^r(\Omega)$, then $g(x,u) \in L^s(\Omega)$. That is

$$g(x,\cdot) : L^r(\Omega) \to L^s(\Omega).$$

Now we verify the continuity of the map. Fix $u \in L^r(\Omega)$. Given any $\epsilon > 0$, we want to show that, there is a $\delta > 0$, such that,

$$\text{whenever } \|\phi\|_{L^r(\Omega)} < \delta, \text{ we have } \|g(\cdot, u+\phi) - g(\cdot, u)\|_{L^s(\Omega)} < \epsilon. \quad (2.32)$$

Let

$$\Omega_1 := \{x \in \bar{\Omega} \mid |\phi(x)| \leq m\}$$

and

$$\Omega_2 := \bar{\Omega} \setminus \Omega_1.$$

Let

$$I_i = \int_{\Omega_i} |g(x, u(x) + \phi(x)) - g(x, u(x))|^s dx, \quad i = 1, 2.$$

By the continuity of $g(x, \cdot)$, for any $\eta > 0$, there exists $m > 0$, such that

$$|g(x, u(x) + \phi(x)) - g(x, u(x))| \leq \eta, \quad \forall x \in \Omega_1.$$

It follows that

$$I_1 \leq |\Omega_1|\eta^s \leq |\Omega|\eta^s,$$

where $|\Omega|$ is the volume of Ω. For the given $\epsilon > 0$, we choose η and then m, so that

$$|\Omega|\eta^s < \left(\frac{\epsilon}{2}\right)^s,$$

and therefore

$$I_1 \leq \left(\frac{\epsilon}{2}\right)^s. \tag{2.33}$$

Now we fix this m, and estimate I_2.

By (g_2), we have

$$I_2 \leq \int_{\Omega_2} (c_1 + c_2(|u|^r + |\phi|^r))\, dx \leq c_1|\Omega_2| + c_2 \int_{\Omega_2} |u|^r dx + c_2\|\phi\|^r_{L^r(\Omega)}, \tag{2.34}$$

for some constant c_1 and c_2.

Moreover, as $\|\phi\|_{L^r(\Omega)} < \delta$,

$$\delta^r > \int_\Omega |\phi|^r dx \geq m^r|\Omega_2|. \tag{2.35}$$

Since $u \in L^r(\Omega)$, we can make $\int_{\Omega_2} |u|^r dx$ as small as we wish by letting $|\Omega_2|$ small. Now by (2.35), we can choose δ so small, such that $|\Omega_2|$ is sufficiently small, and hence the right hand side of (2.34) is less than $\left(\frac{\epsilon}{2}\right)^s$. Consequently, by (2.33), for such a small δ,

$$I_1 + I_2 \leq \left(\frac{\epsilon}{2}\right)^s + \left(\frac{\epsilon}{2}\right)^s.$$

This verifies (2.32), and therefore completes the proof of the Proposition. □

Proposition 2.4.2 *(Rabinowitz [Ra]) Assume that $f(x, s)$ satisfies (f_1) and (f_2). Let $n \geq 3$. Then $I \in C^1(H, R)$ and*

$$< I'(u), v >= \int_\Omega f(x, u)v dx, \quad \forall v \in H.$$

Moreover, $I(\cdot)$ is weakly continuous and $I'(\cdot)$ is compact, i.e.

$$I(u_k) \to I(u) \text{ and } I'(u_k) \to I'(u), \quad \text{whenever } u_k \rightharpoonup u \text{ in } H;$$

Proof. We will first show that I is Frechet differentiable on H and then prove that $I'(u)$ is continuous.

1. Let $u, v \in H$. We want to show that given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, u)$, such that

$$Q(u,v) := |I(u+v) - I(u) - \int_\Omega f(x,u) v dx| \leq \epsilon \|v\| \tag{2.36}$$

whenever $\|v\| \leq \delta$. Here $\|v\|$ denotes the $H^1(\Omega)$ norm of v.

In fact,

$$\begin{aligned} Q(u,v) &\leq \int_\Omega |F(x,u(x)+v(x)) - F(x,u(x)) - f(x,u(x))v(x)| dx \\ &= \int_\Omega |f(x,u(x)+\xi(x)) - f(x,u(x))| \cdot |v(x)| dx \\ &\leq \|f(\cdot, u+\xi) - f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leq \|f(\cdot, u+\xi) - f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)} K\|v\|. \end{aligned} \tag{2.37}$$

Here we have applied the *Mean Value Theorem* (with $|\xi(x)| \leq |v(x)|$), the *Hölder inequality*, and the *Sobolev inequality*

$$\|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq K\|v\|.$$

In Proposition 2.4.1, let $r = \frac{2n}{n-2}$ and $s = \frac{2n}{n+2}$. Then by (f_2), we deduce that the map

$$u(x) \rightarrow f(x, u(x))$$

is continuous from $L^{\frac{2n}{n-2}}(\Omega)$ to $L^{\frac{2n}{n+2}}(\Omega)$. Hence for any given $\epsilon > 0$, we can choose sufficiently small $\delta > 0$, such that whenever $\|v\| < \delta$, we have $\|v\|_{L^{\frac{2n}{n-2}}(\Omega)} < K\delta$, and hence $\|\xi\|_{L^{\frac{2n}{n-2}}(\Omega)} < K\delta$, and therefore

$$\|f(\cdot, u+\xi) - f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)} < \frac{\epsilon}{K}.$$

It follows from (2.37) that

$$Q(u,v) < \epsilon\ , \quad \text{whenever } \|v\| < \delta.$$

This verifies (2.36) and hence $I(\cdot)$ is differentiable, and

$$< I'(u), v >= \int_\Omega f(x,u(x)) v(x) dx. \tag{2.38}$$

2. To verify the continuity of $I'(\cdot)$, we show that for each fixed u,

$$\|I'(u+\phi) - I'(u)\| \rightarrow 0, \quad \text{as } \|\phi\| \rightarrow 0. \tag{2.39}$$

By definition,

$$\begin{aligned}\|I'(u+\phi)-I'(u)\| &= \sup_{\|v\|\leq 1} < I'(u+\phi)-I'(u), v > \\ &= \sup_{\|v\|\leq 1} \int_{\Omega} [f(x,u(x)+\phi(x))-f(x,u(x))]v(x)dx \\ &\leq \sup_{\|v\|\leq 1} \|f(\cdot,u+\phi)-f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)} K\|v\| \\ &\leq K\|f(\cdot,u+\phi)-f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)} \end{aligned} \tag{2.40}$$

Here we have applied (2.38), the *Hölder inequality*, and the *Sobolev inequality*. Now (2.39) follows again from Proposition 2.4.1. Therefore, $I'(\cdot)$ is continuous.

3. Finally, we verify the weak continuity of $I(u)$ and the compactness of $I'(u)$. Assume that

$$\{u_k\} \subset H, \text{ and } u_k \rightharpoonup u \text{ in } H.$$

We show that

$$I(u_k) \rightarrow I(u) \text{ as } k \rightarrow \infty.$$

In fact,

$$\begin{aligned}|I(u_k)-I(u)| &\leq \int_{\Omega} |F(x,u_k(x))-F(x,u(x))|dx \\ &\leq \int_{\Omega} |f(x,\xi_k(x))| \cdot |u_k(x)-u(x)|dx \\ &\leq \|f(\cdot,\xi_k)\|_{L^r(\Omega)} \|u_k-u\|_{L^q(\Omega)} \\ &\leq C\|\xi_k\|_{L^{\frac{2n}{n-2}}(\Omega)} \|u_k-u\|_{L^q(\Omega)} \end{aligned} \tag{2.41}$$

Here we have applied the *Mean Value Theorem* and the *Hölder inequality* with $\xi_k(x)$ between $u_k(x)$ and $u(x)$, and

$$r = \frac{2n}{p(n-2)}, \quad q = \frac{r}{r-1} = \frac{2n}{2n-p(n-2)}.$$

It is easy to see that

$$q < \frac{2n}{n-2}.$$

Since a weak convergent sequence is bounded, $\{u_k\}$ is bounded in H, and hence by *Sobolev Embedding*, it is bounded in $L^{\frac{2n}{n-2}}(\Omega)$, so does $\{\xi_k\}$. Furthermore, by *Compact Embedding* of H into $L^q(\Omega)$, the last term of (2.41) approaches zero as $k \rightarrow \infty$. This verifies the weak continuity of $I(\cdot)$.

Using a similar argument as in deriving (2.40), we obtain

$$\|I'(u_k)-I'(u)\| \leq K\|f(\cdot,u_k)-f(\cdot,u)\|_{L^{\frac{2n}{n+2}}(\Omega)}. \tag{2.42}$$

Since $p < \frac{n+2}{n-2}$, we have $\frac{2np}{n+2} < \frac{2n}{n-2}$. Then by the *Compact Embedding*,

$$H \hookrightarrow\hookrightarrow L^{\frac{2np}{n+2}}(\Omega),$$

we derive

$$u_k \rightarrow u \text{ strongly in } L^{\frac{2np}{n+2}}(\Omega).$$

Consequently, form Proposition 2.4.1,

$$f(\cdot, u_k) \rightarrow f(\cdot, u) \text{ strongly in } L^{\frac{2n}{n+2}}(\Omega).$$

Now, together with (2.42), we arrive at

$$I'(u_k) \rightarrow I'(u) \quad \text{as } k \rightarrow \infty.$$

This completes the proof of the proposition. □

Now let's come back to Problem (2.31) and prove Theorem 2.4.6. We will verify that $J(\cdot)$ satisfies the conditions in the *Mountain Pass Theorem*, and thus possesses a non-trivial mini-max critical point, which is a weak solution of (2.31).

First we verify the (PS) condition. Assume that $\{u_k\}$ is a sequence in H satisfying

$$|J(u_k)| \leq M \text{ and } J'(u_k) \rightarrow 0.$$

We show that $\{u_k\}$ possesses a convergent subsequence.

By (f_4), we have

$$\begin{aligned}
\mu M + \|J'(u_k)\| \|u_k\| &\geq \mu J(u_k) - < J'(u_k), u_k > \\
&= (\frac{\mu}{2} - 1) \int_\Omega |Du_k|^2 dx + \int_\Omega [f(x, u_k) u_k - \mu F(x, u_k)] \, dx \\
&\geq (\frac{\mu}{2} - 1) \int_\Omega |Du_k|^2 dx + \int_{|u_k(x)| \leq r} [f(x, u_k) u_k - \mu F(x, u_k)] \, dx \\
&\geq (\frac{\mu}{2} - 1) \int_\Omega |Du_k|^2 dx - (a_1 r + a_2 r^{p+1}) |\Omega| \\
&= c_o \|u_k\|^2 - C_o, \qquad (2.43)
\end{aligned}$$

for some positive constant c_o and C_o. Since $\|J'(u_k)\|$ is bounded, (2.43) implies that $\{u_k\}$ must be bounded in H. Hence there exists a subsequence (still denoted by $\{u_k\}$), which converges weakly to some element u_o in H.

Let $A : H \rightarrow H^*$ be the duality map between H and its dual space H^*. Then for any $u, v \in H$,

$$< Au, v >= \int_\Omega Du \cdot Dv dx.$$

Consequently,

$$A^{-1}J'(u) = u - A^{-1}f(\cdot, u). \tag{2.44}$$

From Proposition 2.4.2,

$$f(\cdot, u_k) \to f(\cdot, u) \text{ strongly in } H^*.$$

It follows that

$$u_k = A^{-1}J'(u_k) + A^{-1}f(\cdot, u_k) \to A^{-1}f(\cdot, u_o), \quad \text{as } k \to \infty.$$

This verifies the (PS).

To see there is a 'mountain range' surrounding the origin, we estimate $\int_\Omega F(x,u)dx$. By (f_3), for any given $\epsilon > 0$, there is a $\delta > 0$, such that, for all $x \in \bar{\Omega}$,

$$|F(x,s)| \leq \epsilon |s|^2, \text{ whenever } |s| < \delta. \tag{2.45}$$

While by (f_2), there is a constant $M = M(\delta)$, such that for all $x \in \bar{\Omega}$,

$$|F(x,s)| \leq M|s|^{p+1}. \tag{2.46}$$

Combining (2.45) and (2.46), we have,

$$|F(x,s)| \leq \epsilon |s|^2 + M|s|^{p+1}, \quad \text{for all } x \in \bar{\Omega} \text{ and for all } s \in R.$$

It follows, via the *Poincaré* and the *Sobolev inequality*, that

$$|\int_\Omega F(x,u)dx| \leq \epsilon \int_\Omega u^2 dx + M \int_\Omega |u|^{p+1} dx \leq C(\epsilon + M\|u\|^{p-1})\|u\|^2. \tag{2.47}$$

And consequently,

$$J(u) \geq \left[\frac{1}{2} - C(\epsilon + M\|u\|^{p-1})\right] \|u\|^2. \tag{2.48}$$

Choose $\epsilon = \frac{1}{8C}$. For this ϵ, we fix M; then choose $\|u\| = \rho$ small, so that

$$CM\rho^{p-1} \leq \frac{1}{8}.$$

Then form (2.48), we deduce

$$J|_{\partial B_\rho(0)} \geq \frac{1}{4}\rho^2.$$

To see the existence of a point $e \in H$, such that $J(e) \leq 0$, we fix any $u \neq 0$ in H, and consider

$$J(tu) = \frac{t^2}{2}\int_\Omega |Du|^2 dx - \int_\Omega F(x, tu)dx. \tag{2.49}$$

By (f_4), we have for $|s| \geq r$,

$$\frac{d}{ds}\left(|s|^{-\mu}F(x,s)\right) = |s|^{-\mu-2}s\left(-\mu F(x,s) + sf(x,s)\right)\begin{cases} \geq 0 & \text{if } s \geq r \\ \leq 0 & \text{if } s \leq -r. \end{cases}$$

In any case, we deduce, for $|s| \geq r$,

$$F(x,s) \geq a_3|s|^{\mu};$$

and it follows that

$$F(x,s) \geq a_3|s|^{\mu} - a_4. \tag{2.50}$$

Combining (2.49) and (2.50), and taking into account that $\mu > 2$, we conclude that

$$J(tu) \to -\infty, \quad \text{as } t \to \infty.$$

Now J satisfies all the conditions in the *Mountain Pass Theorem*, hence it possesses a minimax critical point u_o, which is a weak solution of (2.31). Moreover, $J(u_o) > 0$, therefore it is a nontrivial solution. This completes the proof of the Theorem 2.4.6. □

3

Regularity of Solutions

In the previous chapter, we used functional analysis, mainly calculus of variations to seek the existence of weak solutions for second order linear or semi-linear elliptic equations. These weak solutions we obtained were in the Sobolev space $W^{1,2}$ and hence, by Sobolev imbedding, in L^p space for $p \leq \frac{n+2}{n-2}$. Usually, the weak solutions are easier to obtain than the classical ones, and this is particularly true for non-linear equations. However, in practice, we are often required to find classical solutions. Therefore, one would naturally want to know whether these weak solutions are actually differentiable, so that they can become classical ones. These questions will be answered here.

In this chapter, we will introduce methods to show that, in most cases, a weak solution is in fact smooth. This is called the regularity argument. Very often, the regularity is equivalent to the a priori estimate of solutions. To see the difference between the *regularity* and the *a priori estimate*, we take the equation

$$-\triangle u = f(x)$$

for example. Roughly speaking, the $W^{2,p}$ regularity theory infers that

If $u \in W^{1,p}$ is a weak solution and if $f \in L^p$, then $u \in W^{2,p}$.

While the $W^{2,p}$ a priori estimate says

If $u \in W_0^{1,p} \cap W^{2,p}$ is a weak solution and if $f \in L^p$, then

$$\|u\|_{W^{2,p}} \leq C\|f\|_{L^p}.$$

In the a priori estimate, we pre-assumed that u is in $W^{2,p}$. It might seem a little bit surprising at this moment that the a priori estimates turn out to be powerful tools in deriving regularities. The readers will see in section 3.2 how the a priori estimates and uniqueness of solutions lead to the regularity.

Let Ω be an open bounded set in R^n. Let $a_{ij}(x), b_i(x)$, and $c(x)$ be bounded functions on Ω with $a_{ij}(x) = a_{ji}(x)$. Consider the second order partial differential equation

$$Lu := -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = f(x). \tag{3.1}$$

We assume L is uniformly elliptic, that is, there exists a constant $\delta > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2$$

for a. e. $x \in \Omega$ and for all $\xi \in R^n$.

In Section 3.1, we establish $W^{2,p}$ a priori estimate for the solutions of (3.1). We show that if the function $f(x)$ is in $L^p(\Omega)$, and $u \in W^{2,p}(\Omega)$ is a strong solution of (3.1), then there exists a constant C, such that

$$\|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|f\|_{L^p}).$$

We will first establish this for a Newtonian potential, then use the method of "frozen coefficients" to generalize it to uniformly elliptic equations.

In Section 3.2, we establish a $W^{2,p}$ regularity. We show that if $f(x)$ is in $L^p(\Omega)$, then any weak solution u of (3.1) is in $W^{2,p}(\Omega)$.

In Section 3.3, we introduce and prove a regularity lifting theorem. It provides a simple method for the study of regularity. The version we present here contains some new developments. It is much more general and very easy to use. We believe that the method will be helpful to both experts and non-experts in the field.

We will also use examples to show how this method can be applied to boost the regularity of the solutions for PDEs as well as for integral equations.

3.1 $W^{2,p}$ a Priori Estimates

We establish $W^{2,p}$ estimate for the solutions of

$$Lu = f(x) \, , \quad x \in \Omega. \tag{3.2}$$

First we start with the Newtonian potential.

3.1.1 Newtonian Potentials

Let Ω be a bounded domain of R^n. Let

$$\Gamma(x) = \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} \quad n \geq 3,$$

be the fundamental solution of the Laplace equation. The Newtonian potential is known as

$$w(x) = \int_\Omega \Gamma(x-y) f(y) dy.$$

We prove

Theorem 3.1.1 *Let $f \in L^p(\Omega)$ for $1 < p < \infty$, and let w be the Newtonian potential of f. Then $w \in W^{2,p}(\Omega)$ and*

$$\triangle w = f(x), \quad a.e. \; x \in \Omega, \tag{3.3}$$

and

$$\|D^2 w\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \tag{3.4}$$

Write $w = Nf$, where N is obviously a linear operator. For fixed $i, \; j$, define the linear operator T as

$$Tf = D_{ij} N f = D_{ij} \int_{R^n} \Gamma(x-y) f(y) dy$$

To prove Theorem 3.1.1, it is equivalent to show that

$$T : L^p(\Omega) \longrightarrow L^p(\Omega) \;\text{ is a bounded linear operator .} \tag{3.5}$$

Our proof can actually be applied to more general operators. To this end, we introduce the concept of weak type and strong type operators.

Define

$$\mu_f(t) = |\{x \in \Omega \mid |f(x)| > t\}|.$$

For $p \geq 1$, a weak L^p space $L^p_w(\Omega)$ is the collection of functions f that satisfy

$$\|f\|^p_{L^p_w(\Omega)} = \sup\{\mu_f(t) t^p, \; \forall t > 0\} < \infty.$$

An operator $T : L^p(\Omega) \to L^q(\Omega)$ is of strong type (p, q) if

$$\|Tf\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \forall f \in L^p(\Omega).$$

T is of weak type (p, q) if

$$\|Tf\|_{L^q_w(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \forall f \in L^p(\Omega).$$

Outline of Proof of (3.5).
We decompose the proof into the proofs of the following four lemmas.
First, we use Fourier transform to show

Lemma 3.1.1 *$T : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator. i.e. T is of strong type $(2, 2)$.*

Secondly, we use Calderon-Zygmund's Decomposition Lemma to prove

Lemma 3.1.2 *T is of weak type (1,1).*

Thirdly, we employ Marcinkiewicz Interpolation Theorem to derive

Lemma 3.1.3 *T is of strong type (r, r) for any $1 < r \leq 2$.*

Finally, by duality, we conclude

Lemma 3.1.4 *T is of strong type (p, p), for $1 < p < \infty$.*

The Proof of Lemma 3.1.1.
First we consider $f \in C_0^\infty(\Omega) \subset C_0^\infty(R^n)$. Then obviously $w \in C^\infty(R^n)$ and satisfies

$$\triangle w = f(x) \ , \ \ \forall x \in R^n.$$

Let

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{i<x,\xi>} f(x) dx,$$

be the Fourier transform of f, where

$$< x, \xi >= \sum_k^n x_k \xi_k$$

is the inner product of R^n, and $i = \sqrt{-1}$.

We need the following simple property of the transform

$$\widehat{D_k} f(\xi) = -i\xi_k \hat{f}(\xi) \ , \quad \text{and } \widehat{D_{jk}} f(\xi) = -\xi_j \xi_k \hat{f}(\xi), \tag{3.6}$$

and the well-known Plancherel's Theorem:

$$\|\hat{f}\|_{L^2(R^n)} = \|f\|_{L^2(R^n)}. \tag{3.7}$$

It follows from (3.6) and (3.7),

$$
\begin{aligned}
\int_\Omega |f(x)|^2 dx &= \int_{R^n} |f(x)|^2 dx \\
&= \int_{R^n} |\triangle w(x)|^2 dx \\
&= \int_{R^n} |\widehat{\triangle w}(\xi)|^2 dx \\
&= \int_{R^n} |\xi|^4 |\hat{w}(\xi)|^2 d\xi \\
&= \sum_{k,j=1}^{n} \int_{R^n} \xi_k^2 \xi_j^2 |\hat{w}(\xi)|^2 d\xi \\
&= \sum_{k,j=1}^{n} \int_{R^n} |\widehat{D_{kj} w}(\xi)|^2 d\xi \\
&= \sum_{k,j=1}^{n} \int_{R^n} |D_{kj} w(x)|^2 dx \\
&= \int_{R^n} |D^2 w|^2 dx.
\end{aligned}
$$

Consequently,

$$\|Tf\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \ , \ \ \forall f \in C_0^\infty(\Omega). \tag{3.8}$$

Now to verify (3.8) for any $f \in L^2(\Omega)$, we simply pick a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ that converges to f in $L^2(\Omega)$, and take the limit. This proves the Lemma. □

The Proof of Lemma 3.1.2.

We need the following well-known Calderón-Zygmund's Decomposition Lemma (see its proof in Appendix C).

Lemma 3.1.5 *For* $f \in L^1(R^n)$, *fixed* $\alpha > 0$, $\exists \quad E,\ G$ *such that*
(i) $R^n = E \cup G,\ E \cap G = \emptyset$
(ii) $|f(x)| \leq \alpha$, *a.e.* $x \in E$
(iii) $G = \bigcup_{k=1}^{\infty} Q_k$, $\{Q_k\}$*: disjoint cubes s.t.*

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} |f(x)| dx \leq 2^n \alpha$$

For any $f \in L^1(\Omega)$, to apply the Calderón-Zygmund's Decomposition Lemma, we first extend f to vanish outside Ω. For any given $\alpha > 0$, fix a large cube Q_o in R^n, such that

$$\int_{Q_o} |f(x)| dx \leq \alpha |Q_o|.$$

We show that

$$\mu_{Tf}(\alpha) := |\{x \in R^n \mid |Tf(x)| \geq \alpha\}| \leq C\frac{\|f\|_{L^1(\Omega)}}{\alpha}. \tag{3.9}$$

Split the function f into the "good" part g and "bad" part b: $f = g + b$, where

$$g(x) = \begin{cases} f(x) & \text{for } x \in E \\ \frac{1}{|Q_k|}\int_{Q_k} f(x)dx & \text{for } x \in Q_k, k = 1, 2, \cdots. \end{cases}$$

Since the operator T is linear, $Tf = Tg + Tb$; and therefore

$$\mu_{Tf}(\alpha) \leq \mu_{Tg}(\frac{\alpha}{2}) + \mu_{Tb}(\frac{\alpha}{2}).$$

We will estimate $\mu_{Tg}(\frac{\alpha}{2})$ and $\mu_{Tb}(\frac{\alpha}{2})$ separately. The estimate of the first one is easy, because $g \in L^2$. To estimate $\mu_{Tb}(\frac{\alpha}{2})$, we divide R^n into two parts: G^* and $E^* := R^n \setminus G^*$ (see below for the precise definition of G^*.) We will show that

(a) $|G^*| \leq \dfrac{C}{\alpha}\|f\|_{L^1(\Omega)}$, and

(b) $|\{x \in E^* \mid |Tb(x)| \geq \dfrac{\alpha}{2}\}| \leq \dfrac{C}{\alpha}\int_{E^*}|Tb(x)|dx \leq \dfrac{C}{\alpha}\|f\|_{L^1(\Omega)}$.

These will imply the desired estimate for $\mu_{Tb}(\frac{\alpha}{2})$.

Obviously, from the definition of g, we have

$$|g(x)| \leq 2^n\alpha \;, \text{ almost everywhere} \tag{3.10}$$

and

$$b(x) = 0 \text{ for } x \in E \;, \text{ and } \int_{Q_k} b(x)dx = 0 \text{ for } k = 1, 2, \cdots. \tag{3.11}$$

We first estimate μ_{Tg}. By Lemma 3.1.1 and (3.10), we derive

$$\mu_{Tg}(\frac{\alpha}{2}) \leq \frac{4}{\alpha^2}\int g^2(x)dx \leq \frac{2^{n+2}}{\alpha}\int |g(x)|dx \leq \frac{2^{n+2}}{\alpha}\int |f(x)|dx. \tag{3.12}$$

We then estimate μ_{Tb}. Let

$$b_k(x) = \begin{cases} b(x) & \text{for } x \in Q_k \\ 0 & \text{elsewhere }. \end{cases}$$

Then

$$Tb = \sum_{k=1}^{\infty} Tb_k.$$

For each fixed k, let $\{b_{km}\} \subset C_0^\infty(Q_k)$ be a sequence converging to b_k in $L^2(\Omega)$ satisfying

$$\int_{Q_k} b_{km}(x)dx = \int_{Q_k} b_k(x)dx = 0. \tag{3.13}$$

From the expression

$$Tb_{km}(x) = \int_{Q_k} D_{ij}\Gamma(x-y)b_{km}(y)dy,$$

one can see that due to the singularity of $D_{ij}\Gamma(x-y)$ in Q_k and the fact that b_{km} may not be bounded in Q_k, one can only estimate $Tb_{km}(x)$ when x is of a positive distance away from Q_k. For this reason, we cover Q_k by a bigger ball B_k which has the same center as Q_k, and the radius of the ball δ_k is the same as the diameter of Q_k. We now estimate the integral in the complement of B_k:

$$\begin{aligned}
\int_{R^n\setminus B_k} |Tb_{km}|(x)dx &= \int_{Q_o\setminus B_k} |\int_{Q_k} D_{ij}\Gamma(x-y)b_{km}(y)dy|dx \\
&= \int_{Q_o\setminus B_k} |\int_{Q_k} [D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\bar{y})]b_{km}(y)dy|dx \\
&\leq C\,\delta_k \int_{Q_o\setminus B_k} \frac{1}{|x|^{n+1}}dx \cdot |\int_{Q_k} b_{km}(y)dy| \\
&\leq C_1\,\delta_k \int_{\delta_k}^{\infty} \frac{1}{r^2}dr \cdot \int_{Q_k} |b_{km}(y)|dy \\
&\leq C_2 \int_{Q_k} |b_{km}(y)|dy \qquad (3.14)
\end{aligned}$$

where $\bar{y}$ is the center of the cube Q_k. One small trick here is to add a term (which is 0 by (3.13)):

$$\int_{Q_k} D_{ij}\Gamma(x-\bar{y})b_{km}(y)dy$$

to produce a helpful factor δ_k by applying the mean value theorem to the difference:

$$D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\bar{y}) = (y-\bar{y})\cdot D(D_{ij}\Gamma)(x-\xi) \leq \delta_k |D(D_{ij})(x-\xi)|.$$

Now letting $m\to\infty$ in (3.14), we obtain

$$\int_{R^n\setminus B_k} |Tb_k(x)|dx \leq C\int_{Q_k} |b_k(y)|dy.$$

Let

$$G^* = \bigcup_{k=1}^{\infty} B_k \quad \text{and} \quad E^* = R^n \setminus G^*.$$

It follows that

$$\begin{aligned}
\int_{E^*} |Tb(x)|dx &\leq C\sum_{k=1}^{\infty}\int_{R^n\setminus G^*} |Tb_k|dx \leq C\sum_{k=1}^{\infty}\int_{R^n\setminus B_k} |Tb_k|dx \\
&\leq C\sum_{k=1}^{\infty}\int_{Q_k} |b_k(y)|dy \leq C\int_{R^n} |f(x)|dx. \qquad (3.15)
\end{aligned}$$

Obviously

$$\mu_{Tb}(\frac{\alpha}{2}) \leq |G^*| + |\{x \in E^* \mid Tb(x) \geq \frac{\alpha}{2}\}|. \tag{3.16}$$

By (iii) in the Calderón-Zygmund's Decomposition Lemma, we have

$$|G^*| = \sum_{k=1}^{\infty} |B_k| = C \sum_{k=1}^{\infty} |Q_k| \leq \frac{C}{\alpha} \sum_{k=1}^{\infty} \int_{Q_k} |f(x)| dx = \frac{C}{\alpha} \int_{R^n} |f(x)| dx. \tag{3.17}$$

Write

$$E_\alpha^* = \{x \in E^* \mid |Tb(x)| \geq \frac{\alpha}{2}\}.$$

Then by (3.15), we derive

$$|E_\alpha^*| \, \frac{\alpha}{2} \leq \int_{E_\alpha^*} |Tb(x)| dx \leq \int_{E^*} |Tb(x)| dx \leq C \int_{R^n} |f(x)| dx. \tag{3.18}$$

Now the desired inequality (3.9) is a direct consequence of (3.12), (3.16), (3.17, and (3.18). This completes the proof of the Lemma.

The proof of Lemma 3.1.3. In the previous lemmas, we have shown that the operator T is of weak type (1, 1) and strong type (2, 2) (of course also weak type (2, 2)). Now Lemma 3.1.3 is a direct consequence of the Marcinkiewicz interpolation theorem in the following restricted form:

Lemma 3.1.6 *Let T be a linear operator from $L^p(\Omega) \cap L^q(\Omega)$ into itself with $1 \leq p < q < \infty$. If T is of weak type (p, p) and weak type (q, q), then for any $p < r < q$, T is of strong type (r, r). More precisely, if there exist constants B_p and B_q, such that*

$$\mu_{Tf}(t) \leq \left(\frac{B_p \|f\|_p}{t}\right)^p \quad \text{and} \quad \mu_{Tf}(t) \leq \left(\frac{B_q \|f\|_q}{t}\right)^q \ , \ \ \forall f \in L^p(\Omega) \cap L^q(\Omega),$$

then

$$\|Tf\|_r \leq C B_p^\theta B_q^{1-\theta} \|f\|_r \ , \ \ \forall f \in L^p(\Omega) \cap L^q(\Omega),$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and C depends only on p, q, and r.

The proof of this lemma can be found in Appendix C.

The proof of Lemma 3.1.4. From the previous lemma, we know that, for any $1 < r \leq 2$, we have

$$\|Tg\|_{L^r(\Omega)} \leq C_r \|g\|_{L^r(\Omega)}. \tag{3.19}$$

Let

$$< f, g >= \int_{\Omega} f(x)g(x)dx$$

be the duality between f and g. Then it is easy to verify that

$$< g, Tf >=< Tg, f > . \tag{3.20}$$

Given any $2 < p < \infty$, let $r = \frac{p}{p-1}$, i.e. $\frac{1}{r} + \frac{1}{p} = 1$. Obviously, $1 < r < 2$. It follows from (3.19) and (3.20) that

$$\begin{aligned}\|Tf\|_{L^p} &= \sup_{\|g\|_{L^r}=1} < g, Tf >= \sup_{\|g\|_{L^r}=1} < Tg, f > \\ &\leq \sup_{\|g\|_{L^r}=1} \|f\|_{L^p}\|Tg\|_{L^r} \leq C_r\|f\|_{L^p}.\end{aligned}$$

This completes the proof of the Lemma.

3.1.2 Uniform Elliptic Equations

In this section, we consider general second order uniform elliptic equations with Dirichlet boundary condition

$$\begin{cases} Lu := -\sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.21}$$

We assume that Ω is bounded with $C^{2,\alpha}$ boundary, $a_{ij}(x) \in C_0(\bar{\Omega})$, $b_i(x) \in L^\infty(\Omega)$, $c(x) \in L^\infty(\Omega)$, and

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$

for some positive constants λ and Λ.

Definition 3.1.1 *We say that u is a strong solution of*

$$Lu = f(x), \quad x \in \Omega$$

if u is twice weakly differentiable in Ω and satisfies the equation almost everywhere in Ω.

Based on the result in the previous section–the estimates on the Newtonian potentials–we will establish a priori estimates on the strong solutions of (3.21).

Theorem 3.1.2 *Let $u \in W^{2,p}(\Omega) \bigcap W_0^{1,2}(\Omega)$ be a strong solution of (3.21). Assume that $f \in L^p(\Omega)$. Then*

$$||u||_{W^{2,p}(\Omega)} \leq C(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}) \tag{3.22}$$

where C is a constant depending on $||b_i(x)||_{L^\infty(\Omega)}$, $||c(x)||_{L^\infty(\Omega)}$, λ, Λ, n, p, and the domain Ω.

Remark 3.1.1 *Conditions*

$$b_i(x), c(x) \in L^\infty(\Omega)$$

can be replaced by the weaker ones

$$b_i(x), c(x) \in L^p(\Omega)\ , \quad \textit{for any } p > n.$$

Proof of Theorem 3.1.2.

We divide the proof into two major parts.

Part 1. Interior Estimate

$$||\nabla^2 u||_{L^p(K)} \leq C(||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}) \tag{3.23}$$

where K is any compact subset of Ω.

Part 2. Boundary Estimate

$$||u||_{W^{2,p}(\Omega\setminus\Omega_\delta)} \leq C(||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}) \tag{3.24}$$

where $\Omega_\delta = \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}$.

Combining the interior and the boundary estimates, we obtain

$$\begin{aligned} ||u||_{W^{2,p}(\Omega)} &\leq ||u||_{W^{2,p}(\Omega\setminus\Omega_{2\delta})} + ||u||_{W^{2,p}(\Omega_\delta)} \\ &\leq C(||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}). \end{aligned} \tag{3.25}$$

Theorem 3.1.2 is then a simple consequence of (3.25) and the following well known Gargaliado-John-Nirenberg type interpolation estimate

$$||\nabla u||_{L^p(\Omega)} \leq C||u||^{1/2}_{L^p(\Omega)}||\nabla^2 u||^{1/2}_{L^p(\Omega)} \leq \epsilon||\nabla^2 u||_{L^p(\Omega)} + \frac{C}{4\epsilon}||u||_{L^p(\Omega)}.$$

Substituting this estimate of $||\nabla u||_{L^p(\Omega)}$ into our inequality (3.25), we get:

$$||u||_{W^{2,p}(\Omega)} \leq C\epsilon||\nabla^2 u||_{L^p(\Omega)} + C(\frac{C}{4\epsilon}||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}).$$

Choosing $\epsilon < \frac{1}{2C}$, we can then absorb the term $||\nabla^2 u||_{L^p(\Omega)}$ on the right hand side by the left hand side and arrive at the conclusion of our theorem

$$||u||_{W^{2,p}(\Omega)} \leq C(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}) \tag{3.26}$$

For both the interior and the boundary estimates, the main idea is the well known method of "frozen" coefficients. Locally, near a point x^o, we may regard the leading coefficients $a_{ij}(x)$ of the operator approximately as the constant $a_{ij}(x^o)$ (as if the functions a_{ij} are frozen at point x^o), and thus we are able to treat the operator as a constant coefficient one and apply the potential estimates derived in the previous section.

Part 1: Interior Estimates.

First we define the cut-off function $\phi(s)$ to be a compact supported C^∞ function:

$$\begin{cases} \phi(s) = 1 & \text{if } s \leq 1 \\ \phi(s) = 0 & \text{if } s \geq 2. \end{cases}$$

Then we quantify the 'module' continuity of a_{ij} with

$$\epsilon(\delta) = \sup_{|x-y|\leq\delta; x,y\in\Omega; 1\leq i,j\leq n} |a_{ij}(x) - a_{ij}(y)|$$

The function $\epsilon(\delta) \searrow 0$ as $\delta \searrow 0$, and it measures the 'module' continuity of the functions a_{ij}.

For any $x^o \in \Omega_{2\delta}$, let

$$\eta(x) = \phi(\frac{|x - x^o|}{\delta}) \text{ and } w(x) = \eta(x)u(x),$$

then

$$\begin{aligned} a_{ij}(x^o)\frac{\partial^2 w}{\partial x_i \partial x_j} &= (a_{ij}(x^o) - a_{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} \\ &= (a_{ij}(x^o) - a_{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \\ &+ a_{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\ &= (a_{ij}(x^o) - a_{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)(b_i(x)\frac{\partial u}{\partial x_i} + c(x)u - f(x)) + \\ &+ a_{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\ &:= F(x) \quad \text{for } x \in R^n. \end{aligned}$$

In the above, we skipped the summation sign $\sum$. We abbreviated $\sum_{i,j=1}^n a_{ij}$ as a_{ij} and so on. Here we notice that all terms are supported in $B_{2\delta} := B_{2\delta}(x^o) \subset \Omega$. With a simple linear transformation, we may assume $a_{ij}(x^o) = \delta_{ij}$. Apparently both w and

$$\Gamma * F := \frac{1}{n(n-2)\omega_n}\int_{R^n} \frac{1}{|x-y|^{n-2}} F(y)dy$$

are solutions of

$$\triangle u = F.$$

Thus by the uniqueness, $w = \Gamma * F$ (see the exercise after the proof of the theorem). Now, following the Newtonian potential estimate (see Theorem 3.3), we obtain:

$$|| \bigtriangledown^2 w||_{L^p(B_{2\delta})} = || \bigtriangledown^2 w||_{L^p(R^n)} \leq C||F||_{L^p(R^n)} = C||F||_{L^p(B_{2\delta})}. \quad (3.27)$$

Calculating term by term, we have

$$||F||_{L^p(B_{2\delta})} \leq \epsilon(2\delta)||\bigtriangledown^2 w||_{L^p(B_{2\delta})} + ||f||_{L^p(B_{2\delta})} + C(||\nabla u||_{L^p(B_{2\delta})} + ||u||_{L^p(B_{2\delta})}).$$

Combining this with the previous inequality (3.27) and choosing δ so small such that $C\epsilon(2\delta) < 1/2$, we get

$$\begin{aligned} || \bigtriangledown^2 w||_{L^p(B_{2\delta})} &\leq C\epsilon(2\delta)|| \bigtriangledown^2 w||_{L^p(B_{2\delta})} + C(||f||_{L^p(B_{2\delta})} + ||\nabla u||_{L^p(B_{2\delta})} + ||u||_{L^p(B_{2\delta})}) \\ &\leq \frac{1}{2}|| \bigtriangledown^2 w||_{L^p(B_{2\delta})} + C(||f||_{L^p(B_{2\delta})} + ||\nabla u||_{L^p(B_{2\delta})} + ||u||_{L^p(B_{2\delta})}). \end{aligned}$$

This is equivalent to

$$|| \bigtriangledown^2 w||_{L^p(B_{2\delta})} \leq C(||f||_{L^p(B_{2\delta})} + ||\nabla u||_{L^p(B_{2\delta})} + ||u||_{L^p(B_{2\delta})})$$

Since $u = w$ on $B_\delta(x^o)$, we have

$$|| \bigtriangledown^2 u||_{L^p(B_\delta)} = || \bigtriangledown^2 w||_{L^p(B_\delta)}.$$

Consequently,

$$|| \bigtriangledown^2 u||_{L^p(B_\delta)} \leq C(||f||_{L^p(B_{2\delta})} + ||\nabla u||_{L^p(B_{2\delta})} + ||u||_{L^p(B_{2\delta})}). \quad (3.28)$$

To extend this estimate from a δ-ball to a compact set, we notice that the collection of balls

$$\{B_\delta(x) \mid x \in \Omega_{2\delta}\}$$

forms an open covering of $\Omega_{2\delta}$. Hence there are finitely many balls $\{B_\delta(x^i) \mid i = 1, ...m\}$ that have already covered $\Omega_{2\delta}$. We now apply the above estimate (3.28) on each of the balls and sum up to obtain

$$\begin{aligned} || \bigtriangledown^2 u||_{L^p(\Omega_{2\delta})} &\leq \sum_{i=1}^{m} || \bigtriangledown^2 u||_{L^p(B_\delta)(x^i)} \\ &\leq \sum_{i=1}^{m} C(||f||_{L^p(B_{2\delta})(x^i)} + ||\nabla u||_{L^p(B_{2\delta})(x^i)} + ||u||_{L^p(B_{2\delta})(x^i)}) \\ &\leq C(||f||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}) \quad (3.29) \end{aligned}$$

For any compact subset K of Ω, let $\delta < \frac{1}{2} dist(K, \partial\Omega)$, then $K \subset \Omega_{2\delta}$, and we derive

$$||\nabla^2 u||_{L^p(K)} \leq C(||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}). \quad (3.30)$$

This establishes the interior estimates.

Part 2. Boundary Estimates

The boundary estimates are very similar. Let's just describe how we can apply almost the same scheme here. For any point $x^o \in \partial\Omega$, $B_\delta(x^o) \bigcap \partial\Omega$

is a $C^{2,\alpha}$ graph for δ small. With a suitable rotation, we may assume that $B_\delta(x^o)\bigcap\partial\Omega$ is given by the graph of

$$x_n = h(x_1, x_2, ..., x_{n-1}) = h(x'),$$

and Ω is on top of this graph locally. Let

$$y = \psi(x) = (x' - x^{o\prime}, x_n - h(x')),$$

then ψ is a diffeomorphism that maps a neighborhood of x^o in Ω onto $B_r^+(0) = \{y \in B_r(0) \mid y_n > 0\}$. The equation becomes

$$\begin{cases} -\sum_{i,j=1}^n \bar{a}_{ij}(y)u_{y_iy_j} + \sum_{i=1}^n \bar{b}_i(y)u_{y_i} + \bar{c}(y)u = \bar{f}(y)\,, \text{ for } y \in B_r^+(0) \\ u(y) = 0\,, \text{ for } y \in \partial B_r^+(0) \text{ with } y_n = 0; \end{cases} \tag{3.31}$$

where the new coefficients are computed from the original ones via the chain rule of differentiation. For example,

$$\bar{a}_{ij}(y) = \frac{\partial \psi^i}{\partial x^l}(\psi^{-1}(y))a_{lk}(\psi^{-1}(y))\frac{\partial \psi^j}{\partial x^k}(\psi^{-1}(y)).$$

If necessary, we make a linear transformation so that $\bar{a}_{ij}(0) = \delta_{ij}$. Since a plane is still mapped to a plane, we may assume that equation (3.31) is valid for some smaller r. Applying the method of frozen coefficients (with $w(y) = \phi(\frac{2|y|}{r})u(y))$ we get:

$$\triangle w(y) = F(y) \text{ on } B_r^+(0).$$

Now let $\bar{w}(y)$ and $\bar{F}(y)$ be the odd extension of $w(y)$ and $F(y)$ from $B_r^+(0)$ to $B_r(0)$, i.e.

$$\bar{w}(y) = \bar{w}(y_1, \cdots, y_{n-1}, y_n) = \begin{cases} w(y_1, \cdots, y_{n-1}, y_n), & \text{if } y_n \geq 0; \\ -w(y_1, \cdots, y_{n-1}, -y_n), & \text{if } y_n < 0. \end{cases}$$

Similarly for $\bar{F}(y)$.

Then one can check that:

$$\triangle \bar{w}(y) = \bar{F}(y)\,, \text{ for } y \in B_r(0).$$

Through the same calculations as in *Part 1*, we derive the basic interior estimate

$$\begin{aligned} \| \bigtriangledown^2 u\|_{L^p(B_r(x^o))} &\leq C(\|f\|_{L^p(B_{2r}(x^o))} + \|\nabla u\|_{L^p(B_{2r}(x^o))} + \|u\|_{L^p(B_{2r}(x^o))}) \\ &\leq C_1(\|f\|_{L^p(B_{2r}(x^o)\cap\Omega)} + \|\nabla u\|_{L^p(B_{2r}(x^o)\cap\Omega)} + \|u\|_{L^p(B_{2r}(x^o)\cap\Omega)}) \end{aligned}$$

for any x^o on $\partial\Omega$ and for some small radius r. The last inequality in the above was due to the symmetric extension of w to $\bar{w}$ from the half ball to the whole ball.

These balls form a covering of the compact set $\partial\Omega$, and thus has a finite covering $B_{r_i}(x^i)$ $(i = 1, 2, ..., k)$. These balls also cover a neighborhood of $\partial\Omega$ including $\Omega\setminus\Omega_\delta$ for some δ small. Summing the estimates up, and following the same steps as we did for the interior estimates, we get:

$$||u||_{W^{2,p}(\Omega\setminus\Omega_\delta)} \leq C(||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}). \tag{3.32}$$

This establishes the boundary estimates and hence completes the proof of the theorem. □

Exercise 3.1.1 *Assume that Ω is bounded, and $w \in W_0^{2,p}(\Omega)$. Show that if $-\triangle w = F$, then $w = \Gamma * F$.*

Hint: (a) Show that $\Gamma * F(x) = 0$ for $x \not\in \bar{\Omega}$.

(b) Show that it is true for $w \in C_0^2(R^n)$.

(c) Show that

$$\triangle(J_\epsilon w) = J_\epsilon(\triangle w)$$

and thus

$$J_\epsilon w = \Gamma * (J_\epsilon F).$$

Let $\epsilon{\rightarrow}0$ to derive $w = \Gamma * F$.

3.2 $W^{2,p}$ Regularity of Solutions

In this section, we will use the a priori estimates established in the previous section to derive the $W^{2,p}$ regularity for the weak solutions.

Let L be a second order uniformly elliptic operator in divergence form

$$Lu = -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u. \tag{3.33}$$

Definition 3.2.1 *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$\begin{cases} Lu = f(x)\,,\ x \in \Omega \\ u(x) = 0\,, \quad x \in \partial\Omega \end{cases} \tag{3.34}$$

if for any $v \in W_0^{1,q}(\Omega)$,

$$\int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}v_{x_j} + \sum_{i}^{n} b_i(x)u_{x_i}v + c(x)uv\right]dx = \int_\Omega f(x)v dx, \tag{3.35}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3.2.1 *Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we only need to require (3.35) be true for all $v \in C_0^\infty(\Omega)$; and this is more convenient in many applications.*

The main result of the section is

Theorem 3.2.1 *Assume Ω is a bounded domain in R^n. Let L be a uniformly elliptic operator defined in (3.33) with $a_{ij}(x)$ Lipschitz continuous and $b_i(x)$ and $c(x)$ bounded.*

Assume that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.34). If $f(x) \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$.

Outline of the Proof. The proof of the theorem is quite complex and will be accomplished in two stages.

In stage 1, we first consider the case when $p \geq 2$, because one can rather easily show the uniqueness of the weak solutions by multiplying both sides of the equation by the solution itself and integrating by parts. As one will see, this uniqueness plays a key role in deriving the regularity.

The proof of the regularity is based on the following fundamental proposition on the existence, uniqueness, and regularity for the solution of the Laplace equation on a unit ball.

Proposition 3.2.1 *Assume $f \in L^p(B_1(0))$ with $p \geq 2$. Then the Dirichlet problem*

$$\begin{cases} \triangle u = f(x) \ , \ x \in B_1(0) \\ u(x) = 0 \ , \quad x \in \partial B_1(0) \end{cases} \tag{3.36}$$

exists a unique solution $u \in W^{2,p}(B_1(0))$ satisfying

$$\|u\|_{W^{2,p}(B_1)} \leq C\|f\|_{L^p(B_1)}. \tag{3.37}$$

We will prove this proposition in subsection 3.2.1, and then use the "frozen" coefficients method to derive the regularity for general operator L.

In stage 2 (subsection 3.2.2), we consider the case $1 < p < 2$. The main difficulty lies in the uniqueness of the weak solution. To show the uniqueness, we first prove a $W^{2,p}$ version of Fredholm Alternative for $p \geq 2$ based on the regularity result in stage 1, which will be used to deduce the existence of solutions for equation

$$-\sum_{ij}(a_{ij}(x)w_{x_i})_{x_j} = F(x).$$

Then we use this equation with a proper choice of $F(x)$ to derive a contradiction if there exists a non-trivial solution of equation

$$-\sum_{ij}(a_{ij}(x)v_{x_i})_{x_j} = 0.$$

After proving the uniqueness, to derive the regularity, everything else is the same as in stage 1.

Remark 3.2.2 *Actually, the conditions on the coefficients of L can be weakened. This will use the Regularity Lifting Theorem in Section 3.3, and hence will be deliberated there.*

3.2.1 The Case $p \geq 2$

We will prove Proposition 3.2.1 and use it to derive the regularity of weak solutions for general operator L. To this end, we need

Lemma 3.2.1 *(*Better A Priori Estimates at Presence of Uniqueness.*) Assume that*

i) for solutions of

$$Lu = f(x)\,, \quad x \in \Omega \tag{3.38}$$

it holds the a priori estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{3.39}$$

and

ii) (uniqueness) if $Lu = 0$, then $u = 0$.

Then we have a better a priori estimate for the solution of (3.38)

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.40}$$

Proof. Suppose the inequality (3.40) is false, then there exists a sequence of functions $\{f_k\}$ with $\|f_k\|_{L^p} = 1$ and the corresponding sequence of solutions $\{u_k\}$ for

$$Lu_k = f_k(x)\,, \quad x \in \Omega$$

such that

$$\|u_k\|_{W^{2,p}} \to \infty \quad \text{as } k \to \infty.$$

It follows from the a priori estimate (3.39) that

$$\|u_k\|_{L^p} \to \infty \quad \text{as } k \to \infty.$$

Let

$$v_k = \frac{u_k}{\|u_k\|_{L^p}} \quad \text{and } g_k = \frac{f_k}{\|u_k\|_{L^p}}.$$

Then

$$\|v_k\|_{L^p} = 1\,, \text{ and } \|g_k\|_{L^p} \to 0. \tag{3.41}$$

From the equation

$$Lv_k = g_k(x)\,, \quad x \in \Omega \tag{3.42}$$

it holds the a priori estimate

$$\|v_k\|_{W^{2,p}} \leq C(\|v_k\|_{L^p} + \|g_k\|_{L^p}). \tag{3.43}$$

(3.41) and (3.43) imply that $\{v_k\}$ is bounded in $W^{2,p}$, and hence there exists a subsequence (still denoted by $\{v_k\}$) that converges weakly to v in $W^{2,p}$. By the compact Sobolev embedding, the same sequence converges strongly to v in L^p, and hence $\|v\|_{L^p} = 1$. From (3.42), we arrive at

$$Lv = 0\ , \quad x \in \Omega.$$

Therefore by the uniqueness assumption, we must have $v = 0$. This contradicts with the fact $\|v\|_{L^p} = 1$ and therefore completes the proof. □

The Proof of Proposition 3.2.1.

For convenience, we abbreviate $B_r(0)$ as B_r.

To see the uniqueness, assume that both u and v are solutions of (3.36). Let $w = u - v$. Then w weakly satisfies

$$\begin{cases} \triangle w = 0\ , & x \in B_1 \\ w(x) = 0\ , & x \in \partial B_1 \end{cases}$$

Multiplying both sides of the equation by w and then integrating on B_1, we derive

$$\int_{B_1} |\bigtriangledown w|^2 dx = 0.$$

Hence $\bigtriangledown w = 0$ almost everywhere, this, together with the zero boundary condition, implies $w = 0$ almost everywhere.

For the existence, it is well-known that, if f is continuous, then

$$u(x) = \int_{B_1} G(x,y) f(y) dy$$

is the solution. Here

$$G(x,y) = \Gamma(x-y) - h(x,y)$$

is the Green's function, in which

$$\Gamma(x) = \begin{cases} \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}\ , & n \geq 3 \\ \frac{1}{2\pi} \ln |x|\ , & n = 2 \end{cases}$$

is the fundamental solution of Laplace equation as introduce in the definition of Newtonian Potentials, and

$$h(x,y) = \frac{1}{n(n-2)\omega_n} \frac{1}{(|x|| \frac{x}{|x|^2} - y|)^{n-2}}\ , \text{ for } n \geq 3$$

is a harmonic function. The addition of $h(x,y)$ is to ensure that $G(x,y)$ vanishes on the boundary.

We have obtained the needed estimate on the first part

$$\int_{B_1} \Gamma(x-y)f(y)dy$$

when we worked on the Newtonian Potentials. For the second part, noticing that $h(x,y)$ is smooth when y is away from the boundary (see the exercise below), we start from a smaller ball $B_{1-\delta}$ for each $\delta > 0$. Let

$$u_\delta(x) = \int_{B_1} G(x,y) f_\delta(y)dy,$$

where

$$f_\delta(x) = \begin{cases} f(x) \ , \ x \in B_{1-\delta} \\ 0 \ , \qquad \text{elsewhere} \ . \end{cases}$$

Now, by our result for the Newtonian Potentials, $D^2 u_\delta$ is in $L^p(B_1)$. Applying the Poincarè inequality, we derive that u_δ is also in $L^p(B_1)$, and hence it is in $W^{2,p}(B_1)$. Moreover, due to uniqueness and Lemma 3.2.1, we have the better a priori estimate

$$\|u_\delta\|_{W^{2,p}(B_1)} \leq C\|f\|_{L^p(B_1)}. \tag{3.44}$$

Pick a sequence $\{\delta_i\}$ tending to zero, then the corresponding solutions $\{u_{\delta_i}\}$ is a Cauchy sequence in $W^{2,p}(B_1)$, because

$$\|u_{\delta_i} - u_{\delta_j}\|_{W^{2,p}(B_1)} \leq C\|f_{\delta_i} - f_{\delta_j}\|_{L^p(B_1)} \to 0 \ , \ \text{as } i,j \to \infty.$$

Let

$$u_o = \lim_{i \to \infty} u_{\delta_i} \ , \ \text{in } W^{2,p}(B_1).$$

Then u_o is a solution of (3.36). Moreover, from 3.44, we see that the better a priori estimate (3.37) holds for u_o.

This completes the proof of Proposition 3.2.1. □

Exercise 3.2.1 *Show that if $|y| \leq 1-\delta$ for some $\delta > 0$, then there exist a constant $c_o > 0$, such that*

$$|x|\left|\frac{x}{|x|^2} - y\right| \geq c_o \ , \forall \, x \in B_1(0).$$

Proof of Regularity Theorem 3.2.1 for $p \geq 2$.

The general frame of the proof is similar to the $W^{2,p}$ a priori estimate in the previous section. The key difference here is that in the a priori estimate, we pre-assume that u is a strong solution in $W^{2,p}$, and here we only assume that u is a weak solution in $W^{1,p}$. After reading the proof of the a priori estimates, the reader may notice that if one can establish a $W^{2,p}$ regularity in a small neighborhood of each point in Ω, then by an entirely similar argument as in obtaining the a priori estimates, one can derive the regularity on the whole of Ω.

As in the proof of the a priori estimates, we define the cut-off function

$$\phi(s) = \begin{cases} 1 \,, & \text{if } s \leq 1 \\ 0 \,, & \text{if } s \geq 2. \end{cases}$$

Let u be a $W^{1,p}_0(\Omega)$ weak solution of (3.34).

For any

$$x^o \in \Omega_{2\delta} := \{x \in \Omega \mid dist(x, \partial\Omega) \geq 2\delta\},$$

let

$$\eta(x) = \phi(\frac{|x - x^o|}{\delta}) \;\text{ and }\; w(x) = \eta(x)u(x).$$

Then w is supported in $B_{2\delta}(x^o)$. By (3.35) and a straight forward calculation, one can verify that, for any $v \in C_0^\infty(B_{2\delta}(x^o))$,

$$\int_{B_{2\delta}(x^o)} a_{ij}(x^o) w_{x_i} v_{x_j} dx = \int_{B_{2\delta}(x^o)} [a_{ij}(x^o) - a_{ij}(x)] w_{x_i} v_{x_j} dx + \int_{B_{2\delta}(x^o)} F(x) v dx$$

where

$$F(x) = f(x) - (a_{ij}(x)\eta_{x_i} u)_{x_j} - b_i(x) u_{x_i} - c(x)u(x).$$

In order words, w is a weak solution of

$$\begin{cases} a_{ij}(x^o) w_{x_i x_j} = ([a_{ij}(x^o) - a_{ij}(x)] w_{x_i})_{x_j} - F(x) \,, & x \in B_{2\delta}(x^o) \\ w(x) = 0 \,, & x \in \partial B_{2\delta}(x^o). \end{cases} \tag{3.45}$$

In the above, we omitted the summation signs $\sum$. We abbreviated $\sum_{i,j=1}^n a_{ij}$ as a_{ij} and so on.

With a change of coordinates, we may assume that $a_{ij}(x^o) = \delta_{ij}$ and write equation (3.45) as

$$\triangle w = ([a_{ij}(x^o) - a_{ij}(x)] w_{x_i})_{x_j} - F(x) \tag{3.46}$$

For any $v \in W^{2,p}(B_{2\delta}(x^o))$, obviously,

$$([a_{ij}(x^o) - a_{ij}(x)] v_{x_i})_{x_j} \in L^p(B_{2\delta}(x^o)).$$

Also, one can easily verify that $F(x)$ is in $L^p(B_{2\delta}(x^o))$. By virtue of Proposition 3.2.1, the operator $\triangle$ is invertible. Consider the equation in $W^{2,p}$

$$v = Kv + \triangle^{-1} F \tag{3.47}$$

where

$$(Kv)(x) = \triangle^{-1} \left([a_{ij}(x^o) - a_{ij}(x)] v_{x_i} \right)_{x_j}.$$

Under the assumption that $a_{ij}(x)$ are Lipschitz continuous, one can verify that (see the exercise below), for sufficiently small δ, K is a contracting map from $W^{2,p}(B_{2\delta}(x^o))$ to itself. Therefore, there exists a unique solution v of equation (3.47). This v is also a weak solution of equation (3.46). Similar to the proof of Proposition 3.2.1, one can show (as an exercise) the uniqueness of the weak solution of (3.46). Therefore, we must have $w = v$ and thus conclude that w is also in $W^{2,p}(B_{2\delta}(x^o))$. This completes the stage 1 of proof for Theorem 3.2.1. □

Exercise 3.2.2 *Assume that each $a_{ij}(x)$ is Lipschitz continuous. Let*

$$(Kv)(x) = \triangle^{-1}\left([a_{ij}(x^o) - a_{ij}(x)]v_{x_i}\right)_{x_j}\ , x \in B_{2\delta}(x^o).$$

Show that, for δ sufficiently small, the operator K is a contracting map in $W^{2,p}(B_{2\delta}(x^o))$, i.e. there exists a constant $0 < \gamma < 1$, such that

$$\|K\phi - K\psi\|_{W^{2,p}(B_{2\delta}(x^o))} \leq \gamma \|\phi - \psi\|_{W^{2,p}(B_{2\delta}(x^o))}.$$

Exercise 3.2.3 *Prove the uniqueness of the $W^{1,p}(B_{2\delta}(x^o))$ weak solution for equation (3.46) when $p \geq 2$.*

3.2.2 The Case $1 < p < 2$.

Lemma 3.2.2 *Let $p \geq 2$. If $u = 0$ whenever $Lu = 0$ (in the sense of $W_0^{1,p}(\Omega)$ weak solution), then for any $f \in L^p(\Omega)$, there exist a unique $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, such that*

$$Lu = f(x) \ \ \text{and} \ \|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p}. \tag{3.48}$$

Proof. This is actually a $W^{2,p}$ version of the *Fredholm Alternative.*

From the previous chapter, one can see that for α sufficiently large, by Lax-Milgram Theorem, for any given $g \in L^2(\Omega)$, there exist a unique $W_0^{1,2}(\Omega)$ weak solution v of the equation

$$(L + \alpha)v = g\ ,\ \ x \in \Omega.$$

From the Regularity Theorem in this chapter, we have $v \in W^{2,2}(\Omega)$. Therefore, we can write

$$v = (L + \alpha)^{-1} g$$

where $(L + \alpha)^{-1}$ is a bounded linear operator from $L^2(\Omega)$ to $W^{2,2}(\Omega)$.

Let $\{f_k\} \subset C_0^\infty(\Omega)$ be a sequence of smooth approximations of $f(x)$. For each k, consider equation

$$(L + \alpha)w - \alpha w = f_k, \tag{3.49}$$

or equivalently,

$$(I - K)w = F, \tag{3.50}$$

where I is the identity operator, $K = \alpha(L+\alpha)^{-1}$, and $F = (L+\alpha)^{-1} f_k$. One can see that K is a compact operator from $W_0^{1,2}(\Omega)$ into itself, because of the compact embedding of $W^{2,2}$ into $W^{1,2}$. Now we can apply the Fredholm Alternative:

Equation (3.50) exists a unique solution if and only if the corresponding homogeneous equation $(I - K)w = 0$ has only trivial solution.

The second half of the above is just the assumption of the theorem, hence, equation (3.50) possesses a unique solution u_k in $W_0^{1,2}(\Omega)$, i.e.

$$u_k - \alpha(L+\alpha)^{-1} u_k = (L+\alpha)^{-1} f_k. \tag{3.51}$$

Furthermore, since both $\alpha(L+\alpha)^{-1}u_k$ and $(L+\alpha)^{-1} f_k$ are in $W^{2,2}(\Omega)$, we derive immediately that u_k is also in $W^{2,2}(\Omega)$, or we can deduce this from the Regularity Theorem 3.2.1. Since f_k is in $L^p(\Omega)$ for any p, we conclude from Theorem 3.2.1 that u_k is in $W^{2,p}(\Omega)$, and furthermore, due to uniqueness, we have the improved a priori estimate (see Lemma 3.2.1)

$$\|u_k\|_{W^{2,p}} \leq C\|f_k\|_{L^p} \ , \quad k = 1, 2, \cdots$$

Moreover, for any integers i and j,

$$L(u_i - u_j) = f_i(x) - f_j(x) \ , \ x \in \Omega.$$

It follows that

$$\|u_i - u_j\|_{W^{2,p}} \leq C\|f_i - f_j\|_{L^p} \to 0 \ , \quad \text{as } i, j \to \infty.$$

This implies that $\{u_k\}$ is a Cauchy sequence in $W^{2,p}$ and hence it converges to an element u in $W^{2,p}(\Omega)$. Now this function u is the desired solution. □

Proposition 3.2.2 (Uniqueness of $W_0^{1,p}$ Weak Solution for $1 < p < \infty$.)

Let Ω be a bounded domain in R^n. Assume that $a_{ij}(x)$ are Lipschitz continuous. If u is a $W_0^{1,p}(\Omega)$ weak solution of

$$(a_{ij}(x) u_{x_i})_{x_j} = 0, \tag{3.52}$$

then $u = 0$ almost every where.

Proof. . When $p \geq 2$, we have proved the uniqueness. Now assume $1 < p < 2$.

Suppose there exist a non-trivial weak solution u. Let $\{u_k\}$ be a sequence of $C_0^\infty(\Omega)$ functions such that

$$u_k \to u \quad \text{in } W_0^{1,p}(\Omega) \ , \quad \text{as } k \to \infty.$$

For each k, consider the equation

$$(a_{ij}(x)v_{x_i})_{x_j} = F_k(x) := \bigtriangledown \cdot \frac{|\bigtriangledown u_k|^{p/q} \bigtriangledown u_k}{\sqrt{1+|\bigtriangledown u_k|^2}}, \tag{3.53}$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Obviously, for $p < 2$, q is greater than 2, and $F_k(x)$ is in $L^q(\Omega)$. Since we already have uniqueness for $W_0^{1,q}$ weak solution, Lemma 3.2.2 guarantees the existence of a solution v_k for equation (3.53).

Through integration by parts several times, one can verify that

$$0 = \int_\Omega v_k (a_{ij}(x)u_{x_i})_{x_j} dx = \int_\Omega \frac{|\bigtriangledown u_k|^{p/q} \bigtriangledown u_k}{\sqrt{1+|\bigtriangledown u_k|^2}} \cdot \bigtriangledown u \, dx. \tag{3.54}$$

Since $u_k \to u$ in $W^{1,p}$, it is easy to see that

$$\frac{|\bigtriangledown u_k|^{p/q} \bigtriangledown u_k}{\sqrt{1+|\bigtriangledown u_k|^2}} \to \frac{|\bigtriangledown u|^{p/q} \bigtriangledown u}{\sqrt{1+|\bigtriangledown u|^2}}$$

in $L^q(\Omega)$, and therefore, by (3.54), we must have $\bigtriangledown u = 0$ almost everywhere. Taking into account that $u \in W_0^{1,p}(\Omega)$, we finally deduce that $u = 0$ almost everywhere. This completes the proof of the proposition. □

After proving the uniqueness, the rest of the arguments are entirely the same as in stage 1. This completes stage 2 of the proof for Theorem 3.2.1.

3.2.3 Other Useful Results Concerning the Existence, Uniqueness, and Regularity

Theorem 3.2.2 *Given any pair $p, q > 1$ and $f \in L^q(\Omega)$, if $u \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$Lu = f(x)\ , \quad x \in \Omega, \tag{3.55}$$

then $u \in W^{2,q}(\Omega)$ and it holds the a priori estimate

$$\|u\|_{W^{2,q}(\Omega)} \leq C(\|u\|_{L^q} + \|f\|_{L^q}). \tag{3.56}$$

Proof. Case i) If $q \leq p$, then obviously, u is also a $W_0^{1,q}(\Omega)$ weak solution, and the results follow from the Regularity Theorem 3.2.1 and the $W^{2,q}$ a priori estimates.

Case ii) If $q > p$, then we use the Sobolev embedding

$$W^{2,p} \hookrightarrow W^{1,p_1},$$

with

$$p_1 = \frac{np}{n-p}.$$

If $p_1 \geq q$, we are done. If $p_1 < q$, we use the fact that u is a W^{1,p_1} weak solution, and by Regularity, $u \in W^{2,p_1}$. Repeating this process, after a few steps, we will arrive at a $p_k \geq q$.

From this Theorem, we can derive immediately the equivalence between the uniqueness of weak solutions in any two spaces $W_0^{1,p}$ and $W_0^{1,q}$:

Corollary 3.2.1 *For any pair $p, q > 1$, the following are equivalent*
i) If $u \in W_0^{1,p}(\Omega)$ is a weak solution of $Lu = 0$, then $u = 0$.
ii) If $u \in W_0^{1,q}(\Omega)$ is a weak solution of $Lu = 0$, then $u = 0$.

Theorem 3.2.3 *(*$W^{2,p}$ Version of the Fredholm Alternative*). Let $1 < p < \infty$. If $u = 0$ whenever $Lu = 0$ (in the sense of $W_0^{1,p}(\Omega)$ weak solution), then for any $f \in L^p(\Omega)$, there exist a unique $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, such that*

$$Lu = f(x) \quad \textit{and} \ \|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p}.$$

For $2 \leq p < \infty$, we have stated this theorem in Lemma 3.2.2. Now, for $1 < p < 2$, after we obtained the $W^{2,p}$ regularity in this case, the proof of this theorem is entirely the same as for Lemma 3.2.2.

3.3 Regularity Lifting

3.3.1 Bootstrap

Assume that $u \in H^1(\Omega)$ is a weak solution of

$$-\triangle u = u^p(x)\ , \quad x \in \Omega \tag{3.57}$$

Then by Sobolev Embedding, $u \in L^{\frac{2n}{n-2}}(\Omega)$. If the power p is less than the critical number $\frac{n+2}{n-2}$, then the regularity of u can be enhanced repeatedly through the equation until we reach that $u \in C^\alpha(\Omega)$. Finally the "Schauder Estimate" will lift u to $C^{2+\alpha}(\Omega)$ and hence to be a classical solution. We call this the "Bootstrap Method" as will be illustrated below.

For each fixed $p < \frac{n+2}{n-2}$, write

$$p = \frac{n+2}{n-2} - \delta$$

for some $\delta > 0$.

Since $u \in L^{\frac{2n}{n-2}}(\Omega)$,

$$u^p \in L^{\frac{2n}{p(n-2)}}(\Omega) = L^{\frac{2n}{(n+2)-\delta(n-2)}}(\Omega).$$

The equation (3.57) boosts the solution u to $W^{2,q_1}(\Omega)$ with

$$q_1 = \frac{2n}{(n+2) - \delta(n-2)}.$$

By the Sobolev Embedding

$$W^{2,q_1}(\Omega) \hookrightarrow L^{\frac{nq_1}{n-2q_1}}(\Omega)$$

we have $u \in L^{s_1}(\Omega)$ with

$$s_1 = \frac{nq_1}{n-2q_1} = \frac{2n}{n-2}\frac{1}{1-\delta}.$$

The integrable power of u has been amplified by $\frac{1}{1-\delta}$ (>1) times.

Now, $u^p \in L^{q_2}(\Omega)$ with

$$q_2 = \frac{s_1}{p} = \frac{4n}{(1-\delta)[n+2-\delta(n-2)]}.$$

And hence through the equation, we derive $u \in W^{2,q_2}(\Omega)$. By Sobolev embedding, if $2q_2 \geq n$, i.e. if

$$(1-\delta)[n+2-\delta(n-2)] - 4 \leq 0\,,$$

then, we have either $u \in L^q(\Omega)$ for any q, or $u \in C^\alpha(\Omega)$. We are done. If $2q_2 < n$, then $u \in L^{s_2}(\Omega)$, with

$$s_2 = \frac{2n}{(1-\delta)[n+2-\delta(n-2)]-4}.$$

It is easy to verify that

$$s_2 \geq \frac{2n}{(1-\delta)(n-2)}\frac{1}{1-\delta} = s_1\frac{1}{1-\delta}.$$

The integrable power of u has again been amplified by at least $\frac{1}{1-\delta}$ times.

Continuing this process, after a finite many steps, we will boost u to $L^q(\Omega)$ for any q, and hence in $C^\alpha(\Omega)$ for some $0 < \alpha < 1$. Finally, by the Schauder estimate, $u \in C^{2+\alpha}(\Omega)$, and therefore, it is a classical solution.

From the above argument, one can see that, if $p = \frac{n+2}{n-2}$, the so called "critical power", then $\delta = 0$, and the integrable power of the solution u can not be boosted this way. To deal with this situation, we introduce a "Regularity Lifting Method" in the next subsection.

3.3.2 Regularity Lifting Theorem

Here, we present a simple method to boost regularity of solutions. It has been used extensively in various forms in the authors previous works. The essence of the approach is well-known in the analysis community. However, the version we present here contains some new developments. It is much more general and is very easy to use. We believe that our method will provide convenient ways, for both experts and non-experts in the field, in obtaining regularities. Essentially, it is based on the following Regularity Lifting Theorem.

Let Z be a given vector space. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be two norms on Z. Define a new norm $\|\cdot\|_Z$ by

$$\|\cdot\|_Z = \sqrt[p]{\|\cdot\|_X^p + \|\cdot\|_Y^p}$$

For simplicity, we assume that Z is complete with respect to the norm $\|\cdot\|_Z$. Let X and Y be the completion of Z under $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Here, one can choose p, $1 \leq p \leq \infty$, according to what one needs. It's easy to see that $Z = X \cap Y$.

Theorem 3.3.1 *Let T be a contraction map from X into itself and from Y into itself. Assume that $f \in X$, and that there exits a function $g \in Z$ such that $f = Tf + g$. Then f also belongs to Z.*

Proof. Step 1. First show that $T : Z \longrightarrow Z$ is a contraction. Since T is a contraction on X, there exists a constant $\theta_1,\ 0 < \theta_1 < 1$ such that

$$\|Th_1 - Th_2\|_X \leq \theta_1 \|h_1 - h_2\|_X.$$

Similarly, we can find a constant $\theta_2,\ 0 < \theta_2 < 1$ such that

$$\|Th_1 - Th_2\|_Y \leq \theta_2 \|h_1 - h_2\|_Y.$$

Let $\theta = \max\{\theta_1, \theta_2\}$. Then, for any $h_1,\ h_2 \in Z$,

$$\begin{aligned} \|Th_1 - Th_2\|_Z &= \sqrt[p]{\|Th_1 - Th_2\|_X^p + \|Th_1 - Th_2\|_Y^p} \\ &\leq \sqrt[p]{\theta_1^p \|h_1 - h_2\|_X^p + \theta_2^p \|h_1 - h_2\|_Y^p} \\ &\leq \theta \|h_1 - h_2\|_Z. \end{aligned}$$

Step 2. Since $T : Z \longrightarrow Z$ is a contraction, given $g \in Z$, we can find a solution $h \in Z$ such that $h = Th + g$. We see that $T : X \longrightarrow X$ is also a contraction and $g \in Z \subset X$. The equation $x = Tx + g$ has a unique solution in X. Thus, $f = h \in Z$ since both h and f are solutions of $x = Tx + g$ in X.

3.3.3 Applications to PDEs

Now, we explain how the "Regularity Lifting Theorem" proved in the previous subsection can be used to boost the regularity of week solutions involving critical exponent:

$$-\triangle u = u^{\frac{n+2}{n-2}}. \tag{3.58}$$

Still assume that Ω is a smooth bounded domain in R^n with $n \geq 3$. Let $u \in H_0^1(\Omega)$ be a weak solution of equation (3.58). Then by Sobolev embedding,

$$u \in L^{\frac{2n}{n-2}}(\Omega).$$

We can split the right hand side of (3.58) in two parts:

$$u^{\frac{n+2}{n-2}} = u^{\frac{4}{n-2}} u := a(x)u.$$

Then obviously, $a(x) \in L^{\frac{n}{2}}(\Omega)$. Hence, more generally, we consider the regularity of the weak solution of the following equation

$$-\triangle u = a(x)u + b(x). \tag{3.59}$$

Theorem 3.3.2 *Assume that $a(x), b(x) \in L^{\frac{n}{2}}(\Omega)$. Let u be any weak solution of equation (3.59) in $H_0^1(\Omega)$. Then u is in $L^p(\Omega)$ for any $1 \leq p < \infty$.*

Remark 3.3.1 *Even in the best situation when $a(x) \equiv 0$, we can only have $u \in W^{2,\frac{n}{2}}(\Omega)$ via the $W^{2,p}$ estimate, and hence derive that $u \in L^p(\Omega)$ for any $p < \infty$ by Sobolev embedding.*

Now let's come back to nonlinear equation (3.58). Assume that u is a $H_0^1(\Omega)$ weak solution. From the above theorem, we first conclude that u is in $L^q(\Omega)$ for any $1 < q < \infty$. Then by our Regularity Theorem 3.2.1, u is in $W^{2,q}(\Omega)$. This implies that $u \in C^{1,\alpha}(\Omega)$ via Sobolev embedding. Finally, from the classical $C^{2,\alpha}$ regularity, we arrive at $u \in C^{2,\alpha}(\Omega)$.

Corollary 3.3.1 *If u is a $H_0^1(\Omega)$ weak solution of equation (3.58), then $u \in C^{2,\alpha}(\Omega)$, and hence it is a classical solution.*

Proof of Theorem 3.3.2.

Let $G(x,y)$ be the Green's function of $-\triangle$ in Ω. Define

$$(Tf)(x) = (-\triangle)^{-1} f(x) = \int_\Omega G(x,y) f(y) dy.$$

Then obviously

$$0 < G(x,y) < \Gamma(|x-y|) := \frac{C_n}{|x-y|^{n-2}},$$

and it follows that, for any $q \in (1, \frac{n}{2})$,

$$\|Tf\|_{L^{\frac{nq}{n-2q}}(\Omega)} \leq \|\Gamma * f\|_{L^{\frac{nq}{n-2q}}(R^n)} \leq C\|f\|_{L^q(\Omega)}.$$

Here, we may extend f to be zero outside Ω when necessary.

For a positive number A, define

$$a_A(x) = \begin{cases} a(x) & \text{if } |a(x)| \geq A \\ 0 & \text{otherwise} \end{cases},$$

and

$$a_B(x) = a(x) - a_A(x).$$

Let

$$(T_A u)(x) = \int_\Omega G(x,y) a_A(y) u(y) dy.$$

Then equation (3.59) can be written as

$$u(x) = (T_A u)(x) + F_A(x),$$

where

$$F_A(x) = \int_\Omega G(x,y)[a_B(y)u(y) + b(y)]dy.$$

We will show that, for any $1 \leq p < \infty$,

i) T_A is a contracting operator from $L^p(\Omega)$ to $L^p(\Omega)$ for A large, and

ii) $F_A(x)$ is in $L^p(\Omega)$.

Then by the "Regularity Lifting Theorem", we derive immediately that $u \in L^p(\Omega)$.

i) *The Estimate of the Operator* T_A.

For any $\frac{n}{n-2} < p < \infty$, there is a q, $1 < q < \frac{n}{2}$, such that

$$p = \frac{nq}{n-2q}$$

and then by Hölder inequality, we derive

$$\|T_A u\|_{L^p(\Omega)} \leq \|\Gamma * a_A u\|_{L^p(R^n)} \leq C\|a_A u\|_{L^q(\Omega)} \leq C\|a_A\|_{L^{\frac{n}{2}}(\Omega)} \|u\|_{L^p(\Omega)}.$$

Since $a(x) \in L^{\frac{n}{2}}(\Omega)$, one can choose a large number A, such that

$$\|a_A\|_{L^{\frac{n}{2}}(\Omega)} \leq \frac{1}{2}$$

and hence arrive at

$$\|T_A u\|_{L^p(\Omega)} \leq \frac{1}{2}\|u\|_{L^p(\Omega)}.$$

That is $T_A : L^p(\Omega) \rightarrow L^p(\Omega)$ is a contracting operator.

ii) *The Integrability of* $F_A(x)$.

(a) First consider

$$F_A^1(x) := \int_\Omega G(x,y) b(y) dy.$$

For any $\frac{n}{n-2} < p < \infty$, choose $1 < q < \frac{n}{2}$, such that $p = \frac{nq}{n-2q}$. Extending $b(x)$ to be zero outside Ω, we have

$$\|F_A^1\|_{L^p(\Omega)} \leq \|\Gamma * b\|_{L^p(R^n)} \leq C\|b\|_{L^q(\Omega)} \leq C\|b\|_{L^{\frac{n}{2}}(\Omega)} < \infty.$$

Hence

$$F_A^1(x) \in L^p(\Omega).$$

(b) Then estimate

$$F_A^2(x) = \int_\Omega G(x,y)a_B(y)u(y)dy.$$

By the bounded-ness of $a_B(x)$, we have

$$\|F_A^2\|_{L^p(\Omega)} \leq \|a_B u\|_{L^q(\Omega)} \leq C\|u\|_{L^q(\Omega)}.$$

Noticing that $u \in L^q(\Omega)$ for any $1 < q \leq \frac{2n}{n-2}$, and

$$p = \frac{2n}{n-6} \quad \text{when } q = \frac{2n}{n-2},$$

we conclude that, for the following values of p and dimension n,

$$\begin{cases} 1 < p < \infty & \text{when } 3 \leq n \leq 6 \\ 1 < p < \frac{2n}{n-6} & \text{when } n > 6, \end{cases}$$

$F_A(x) \in L^p(\Omega)$, and hence T_A is a contracting operator from $L^p(\Omega)$ to $L^p(\Omega)$.

Applying the *Regularity Lifting Theorem*, we arrive at

$$\begin{cases} u \in L^p(\Omega)\,, & \text{for any } 1 < p < \infty & \text{if } 3 \leq n \leq 6 \\ u \in L^p(\Omega)\,, & \text{for any } 1 < p < \frac{2n}{n-6} & \text{if } n > 6. \end{cases}$$

The only thing that prevent us from reaching the full range of p is the estimate of $F_A^2(x)$. However, from the starting point where $u \in L^{\frac{2n}{n-2}}(\Omega)$, we have arrived at

$$u \in L^p(\Omega)\,, \quad \text{for any } 1 < p < \frac{2n}{n-6}.$$

Now from this point on, through an entire similar argument as above, we will reach

$$\begin{cases} u \in L^p(\Omega)\,, & \text{for any } 1 < p < \infty & \text{if } 3 \leq n \leq 10 \\ u \in L^p(\Omega)\,, & \text{for any } 1 < p < \frac{2n}{n-10} & \text{if } n > 10. \end{cases}$$

Each step, we add 4 to the dimension n for which holds

$$u \in L^p(\Omega)\,,\ 1 \leq p < \infty.$$

Continuing this way, we prove our theorem for all dimensions. This completes the proof of the Theorem. □

Now we will use the similar idea and the Regularity Lifting Theorem to carry out Stage 3 of the proof of the Regularity Theorem.

Let L be a second order uniformly elliptic operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u. \tag{3.60}$$

we will prove

Theorem 3.3.3 *($W^{2,p}$ Regularity under Weaker Conditions)*
Let Ω be a bounded domain in R^n and $1 < p < \infty$. Assume that
i) $a_{ij}(x) \in W^{1,n+\delta}(\Omega)$ for some $\delta > 0$;
ii)

$$b_i(x) \in \begin{cases} L^n(\Omega), & \text{if } 1 < p < n \\ L^{n+\delta}(\Omega), & \text{if } p = n \\ L^p(\Omega), & \text{if } n < p < \infty; \end{cases}$$

iii)

$$c(x) \in \begin{cases} L^{n/2}(\Omega), & \text{if } 1 < p < n/2 \\ L^{n/2+\delta}(\Omega), & \text{if } p = n/2 \\ L^p(\Omega), & \text{if } n/2 < p < \infty. \end{cases}$$

Suppose $f \in L^p(\Omega)$ and u is a $W_0^{1,p}(\Omega)$ weak solution of

$$Lu = f(x) \quad x \in \Omega. \tag{3.61}$$

Then u is in $W^{2,p}(\Omega)$.

Proof. Write equation (3.61) as

$$-\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = -\left(\sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u\right) + f(x).$$

Let

$$L_o u := -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j}.$$

By Proposition 3.2.2, we know that equation $L_o u = 0$ has only trivial solution, and it follows from Theorem 3.2.3, the operator L_o is invertible. Denote it by L_o^{-1}. Then L_o^{-1} is a bounded linear operator from $L^p(\Omega)$ to $W^{2,p}(\Omega)$.

For each i and each positive number A, define

$$b_{iA}(x) = \begin{cases} b_i(x) \,, & \text{if } |b_i(x)| \geq A \\ 0 \,, & \text{elsewhere} \,. \end{cases}$$

Let

$$b_{iB}(x) := b_i(x) - b_{iA}(x).$$

Similarly for $C_A(x)$ and $C_B(x)$.

Now we can rewrite equation (3.61) as

$$u = T_A u + F_A(u, x) \tag{3.62}$$

where

$$T_A u = -L_o^{-1}\left(\sum_i b_{iA}(x)u_{x_i} + c_A(x)u\right)$$

and

$$F_A(u,x) = -L_o^{-1}\left(\sum_i b_{iB}(x)u_{x_i} + c_B(x)u\right) + L_o^{-1}f(x).$$

We first show that

$$F_A(u,\cdot) \in W^{2,p}(\Omega)\ ,\ \forall u \in W^{1,p}(\Omega), f \in L^p(\Omega). \tag{3.63}$$

Since $b_{iB}(x)$ and $c_{iB}(x)$ are bounded, one can easily verify that

$$\left(\sum_i b_{iB}(x)u_{x_i} + c_B(x)u\right) \in L^p(\Omega)$$

for any $u \in W^{1,p}(\Omega)$. This implies (3.63).

Then we prove that T_A is a contracting operator from $W^{2,p}(\Omega)$ to $W^{2,p}(\Omega)$.

In fact, by Hölder inequality and Sobolev embedding from $W^{2,p}$ to $W^{1,q}$, one can see that, for any $v \in W^{2,p}(\Omega)$,

$$\|b_{iA}Dv\|_{L^p} \leq C\cdot\begin{cases}\|b_{iA}\|_{L^n}\|v\|_{W^{2,p}}\ , & \text{if } 1<p<n\\ \|b_{iA}\|_{L^{n+\delta}}\|v\|_{W^{2,p}}\ , & \text{if } p=n\\ \|b_{iA}\|_{L^p}\|v\|_{W^{2,p}}\ , & \text{if } n<p<\infty,\end{cases} \tag{3.64}$$

and

$$\|c_A v\|_{L^p} \leq C\cdot\begin{cases}\|c_A\|_{L^{n/2}}\|v\|_{W^{2,p}}\ , & \text{if } 1<p<n/2\\ \|c_A\|_{L^{n/2+\delta}}\|v\|_{W^{2,p}}\ , & \text{if } p=n/2\\ \|c_A\|_{L^p}\|v\|_{W^{2,p}}\ , & \text{if } n/2<p<\infty.\end{cases} \tag{3.65}$$

Under the conditions of the theorem, the first norms on the right hand side of (3.64) and (3.65), i.e. $\|b_{iA}\|_{L^n}$, $\|c_A\|_{L^{n/2}}$ can be made arbitrarily small if A is sufficiently large. Hence for any $\epsilon > 0$, there exists an $A > 0$, such that

$$\|\sum_i b_{iA}(x)v_{x_i} + c_A(x)v\|_{L^p} \leq \epsilon\|v\|_{W^{2,p}}\ ,\ \forall\, v \in W^{2,p}(\Omega).$$

Taking into account that L_o^{-1} is a bounded operator from $L^p(\Omega)$ to $W^{2,p}(\Omega)$, we deduce that T_A is a contract mapping from $W^{2,p}(\Omega)$ to $W^{2,p}(\Omega)$. It follows from the Contract Mapping Theorem, there exist a $v \in W^{2,p}(\Omega)$, such that

$$v = T_A v + F_A(u,x).$$

Now u and v satisfy the same equation. By uniqueness, we must have $u = v$. Therefore we derive that $u \in W^{2,p}(\Omega)$. This completes the proof of the theorem. □

3.3.4 Applications to Integral Equations

As another application of the *Regularity Lifting Theorem*, we consider the following integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy \, , x \in R^n, \tag{3.66}$$

where α is any real number satisfying $0 < \alpha < n$.

It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities.

It is also closely related to the following family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)} \, , x \in R^n \; n \geq 3. \tag{3.67}$$

Actually, one can prove (See [CLO]) that the integral equation (3.66) and the PDE (3.67) are equivalent.

In the special case $\alpha = 2$, (3.67) becomes

$$-\Delta u = u^{(n+2)/(n-2)} \, , x \in R^n .$$

which is the well known Yamabe equation.

In the context of the Hardy-Littlewood-Sobolev inequality, it is natural to start with $u \in L^{\frac{2n}{n-\alpha}}(R^n)$. We will use the *Regularity Lifting Theorem* to boost u to $L^q(R^n)$ for any $1 < q < \infty$, and hence in $L^\infty(R^n)$. More generally, we consider the following equation

$$u(x) = \int_{R^n} \frac{K(y)|u(y)|^{p-1}u(y)}{|x-y|^{n-\alpha}} dy \tag{3.68}$$

for some $1 < p < \infty$. We prove

Theorem 3.3.4 *Let u be a solution of (3.68). Assume that*

$$u \in L^{q_o}(R^n) \;\; \textit{for some } q_o > \frac{n}{n-\alpha} \, , \tag{3.69}$$

$$\int_{R^n} |K(y)|u(y)|^{p-1}|^{\frac{n}{\alpha}} dy < \infty \, , \;\; \textit{and } |K(y)| \leq C(1 + \frac{1}{|y|^\gamma}) \;\; \textit{for some } \gamma < \alpha \, . \tag{3.70}$$

Then u is in $L^q(R^n)$ for any $1 < q < \infty$.

Remark 3.3.2 *i) If*

$$p > \frac{n}{n-\alpha}, \; |K(x)| \leq M, \;\; \textit{and } u \in L^{\frac{n(p-1)}{\alpha}}(R^n), \tag{3.71}$$

then conditions (3.69) and (3.70) are satisfied.

ii) Last condition in (3.71) is somewhat sharp in the sense that if it is violated, then equation (3.68) when $K(x) \equiv 1$ possesses singular solutions such as

$$u(x) = \frac{c}{|x|^{\frac{\alpha}{p-1}}}.$$

Proof of Theorem 3.3.4: Define the linear operator

$$T_v w = \int_{R^n} \frac{K(y)|v(y)|^{p-1} w}{|x-y|^{n-\alpha}} dy.$$

For any real number $a > 0$, define

$$\begin{cases} u_a(x) = u(x) \,, & \text{if } |u(x)| > a \,, \text{ or if } |x| > a \\ u_a(x) = 0 \,, & \text{otherwise} \,. \end{cases}$$

Let $u_b(x) = u(x) - u_a(x)$.

Then since u satisfies equation (3.68), one can verify that u_a satisfies the equation

$$u_a = T_{u_a} u_a + g(x) \tag{3.72}$$

with the function

$$g(x) = \int_{R^n} \frac{K(y)|u_b(y)|^{p-1} u_b(y)}{|x-y|^{n-\alpha}} dy - u_b(x).$$

Under the second part of the condition (3.70), it is obvious that

$$g(x) \in L^\infty \cap L^{q_o}.$$

For any $q > \frac{n}{n-\alpha}$, we first apply the Hardy-Littlewood-Sobolev inequality to obtain

$$\|T_{u_a} w\|_{L^q} \leq C(\alpha, n, q) \|K|u_a|^{p-1} w\|_{L^{\frac{nq}{n+\alpha q}}}. \tag{3.73}$$

Then write $H(x) = K(x)|u_a(x)|^{p-1}$, we apply the Hölder inequality to the right hand side of the above inequality

$$\begin{aligned} &\left\{ \int_{R^n} |H(y)|^{\frac{nq}{n+\alpha q}} |f(y)|^{\frac{nq}{n+\alpha q}} dy \right\}^{\frac{n+\alpha q}{nq}} \\ &\leq \left\{ \left(\int_{R^n} |H(y)|^{\frac{nq}{n+\alpha q} \cdot r} dy \right)^{\frac{1}{r}} \left(\int_{R^n} |f(y)|^{\frac{nq}{n+\alpha q} \cdot s} dy \right)^{\frac{1}{s}} \right\}^{\frac{n+\alpha q}{nq}} \\ &= \left(\int_{R^n} |H(y)|^{\frac{n}{\alpha}} dy \right)^{\frac{\alpha}{n}} \|w\|_{L^q}. \end{aligned} \tag{3.74}$$

Here we have chosen

$$s = \frac{n+\alpha q}{n} \quad \text{and } r = \frac{n+\alpha q}{\alpha q}.$$

It follows from (3.73) and (3.74) that

$$\|T_{u_a} w\|_{L^q} \leq C(\alpha, n, q)(\int |K(y)|u_a(y)|^{p-1}|^{\frac{n}{\alpha}} \, dy)^{\frac{\alpha}{n}} \|w\|_{L^q}. \tag{3.75}$$

By (3.70) and (3.75), we deduce, for sufficiently large a,

$$\|T_{u_a} w\|_{L^q} \leq \frac{1}{2} \|w\|_{L^q}. \tag{3.76}$$

Applying (3.76) to both the case $q = q_o$ and the case $q = p_o > q_o$, and by the *Contracting Mapping Theorem*, we see that the equation

$$w = T_{u_a} w + g(x) \tag{3.77}$$

has a unique solution in both L^{q_o} and $L^{p_o} \cap L^{q_o}$. From (3.72), u_a is a solution of (3.77) in L^{q_o}. Let w be the solution of (3.77) in $L^{p_o} \cap L^{q_o}$, then w is also a solution in L^{q_o}. By the uniqueness, we must have $u_a = w \in L^{p_o} \cap L^{q_o}$ for any $p_o > \frac{n}{n-\alpha}$. So does u. This completes the proof of the Theorem.

4

Preliminary Analysis on Riemannian Manifolds

According to Einstein, the Universe we live in is a curved space. We call this curved space a manifold, which is a generalization of the flat Euclidean space. Roughly speaking, each neighborhood of a point on a manifold is homeomorphic to an open set in the Euclidean space, and the manifold as a whole is obtained by pasting together pieces of the Euclidean spaces. In this section, we will introduce the most basic concepts of differentiable manifolds, tangent space, Riemannian metrics, and curvatures. We will try to be as intuitive and brief as possible. For more rigorous arguments and detailed discussions, please see, for example, the book *Riemannian Geometry* by do Carmo [Ca].

4.1 Differentiable Manifolds

A simple example of a two dimensional differentiable manifold is a regular surface in R^3. Intuitively, it is a union of open sets of R^2, organized in such a way that at the intersection of any two open sets the change from one to the other can be made in a differentiable manner. More precisely, we have

Definition 4.1.1 *(Differentiable Manifolds)*

A differentiable n-dimensional manifold M is

(i) a union of open sets, $M = \bigcup_\alpha V_\alpha$

and a family of homeomorphisms ϕ_α from V_α to an open set $\phi_\alpha(V_\alpha)$ in R^n, such that

(ii) For any pair α and β with $V_\alpha \bigcap V_\beta = U \neq \emptyset$, the images of the intersection $\phi_\alpha(U)$ and $\phi_\beta(U)$ are open sets in R^n and the mappings $\phi_\beta \circ \phi_\alpha^{-1}$ are differentiable.

(iii) The family $\{V_\alpha, \phi_\alpha\}$ are maximal relative to the conditions (i) and (ii).

A manifold M is a union of open sets. In each open set $U \subset M$, one can define a coordinates chart. In fact, let ϕ_U be the homeomorphism from U to $\phi_U(U)$ in R^n, for each $p \in U$, if $\phi_U(p) = (x_1, x_2, \cdots, x_n)$ in R^n, then we can define the coordinates of p to be $(x_1, x_2, \cdots x_n)$. This is called a local coordinates of p, and (U, ϕ_U) is called a coordinates chart. If a family of coordinates charts that covers M are well made so that the transition from one to the other is in a differentiable way as in condition (ii), then it is called a differentiable structure of M. Given a differentiable structure on M, one can easily complete it into a maximal one, by taking the union of all the coordinates charts of M that are compatible with (in the sense of condition (ii)) the ones in the given structure.

A trivial example of n-dimensional manifolds is the Euclidean space R^n, with the differentiable structure given by the identity. The following are two non-trivial examples.

Example 1. The n-dimensional unit sphere

$$S^n = \{x \in R^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Take the unit circle S^1 for instance, we can use the following four coordinates charts:

$$\begin{cases} V_1 = \{x \in S^1 \mid x_2 > 0\}, \phi_1(x) = x_1, \\ V_2 = \{x \in S^1 \mid x_2 < 0\}, \phi_2(x) = x_1, \\ V_3 = \{x \in S^1 \mid x_1 > 0\}, \phi_3(x) = x_2, \\ V_4 = \{x \in S^1 \mid x_1 < 0\}, \phi_4(x) = x_2. \end{cases}$$

Obviously, S^1 is the union of the four open sets V_1, V_2, V_3, and V_4. On the intersection of any two open sets, say on $V_1 \bigcap V_3$, we have

$$\begin{cases} x_2 = \sqrt{1 - x_1^2}, & x_1 > 0; \\ x_1 = \sqrt{1 - x_2^2}, & x_2 > 0. \end{cases}$$

They are both differentiable functions. Same is true on other intersections. Therefore, with this differentiable structure, S^1 is a 1-dimensional differentiable manifold.

Example 2. The n-dimensional real projective space $P^n(R)$. It is the set of straight lines of R^{n+1} which passes through the origin $(0, \cdots, 0) \in R^{n+1}$, or the quotient space of $R^{n+1} \setminus \{0\}$ by the equivalence relation

$$(x_1, \cdots x_{n+1}) \sim (\lambda x_1, \cdots, \lambda x_{n+1}),\ \lambda \in R,\ \lambda \neq 0.$$

We denote the points in $P^n(R)$ by $[x]$. Observe that if $x_i \neq 0$, then

$$[x_1, \cdots, x_{n+1}] = [x_1/x_i, \cdots x_{i-1}/x_i, 1, x_{i+1}/x_i, \cdots x_{n+1}/x_i].$$

For $i = 1, 2, \cdots, n+1$, let

$$V_i = \{[x] \mid x_i \neq 0\}.$$

Apparently,

$$P^n(R) = \bigcup_{i=1}^{n+1} V_i.$$

Geometrically, V_i is the set of straight lines of R^{n+1} which passes through the origin and which do not belong to the hyperplane $x_i = 0$.

Define

$$\phi_i([x]) = (\xi_1^i, \cdots, \xi_{i-1}^i, \xi_{i+1}^i, \cdots, \xi_{n+1}^i),$$

where $\xi_k^i = x_k/x_i$ $(i \neq k)$ and $1 \leq i \leq n+1$.

On the intersection $V_i \bigcap V_j$, the changes of coordinates

$$\begin{cases} \xi_k^j = \frac{\xi_k^i}{\xi_j^i} & k \neq i, j \\ \xi_i^j = \frac{1}{\xi_j^i}. \end{cases}$$

are obviously smooth, thus $\{V_i, \phi_i\}$ is a differentiable structure of $P^n(R)$ and hence $P^n(R)$ is a differentiable manifold.

4.2 Tangent Spaces

We know that at every point of a regular curve or at a regular surface, there exists a tangent line or a tangent plane. Similarly, given a differentiable structure on a manifold, we can define a tangent space at each point. For a regular surfaces S in R^3, at each given point p, the tangent plane is the space of all tangent vectors, and each tangent vector is defined as the "velocity in the ambient space R^3. Since on a differential manifold, there is no such an ambient space, we have to find a characteristic property of the tangent vector which will not involve the concepts of velocity. To this end, let's consider a smooth curve in R^n:

$$\gamma(t) = (x_1(t), \cdots, x_n(t)),\ \ t \in (-\epsilon, \epsilon), \quad \text{with } \gamma(0) = p.$$

The tangent vector of γ at point p is

$$v = (x_1'(0), \cdots, x_n'(0)).$$

Let $f(x)$ be a smooth function near p. Then the directional derivative of f along vector $v \in R^n$ can be expressed as

$$\frac{d}{dt}(f \circ \gamma) \mid_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \frac{\partial x_i}{\partial t}(0) = (\sum_i x_i'(0) \frac{\partial}{\partial x_i}) f.$$

Hence the directional derivative is an operator on differentiable functions that depends uniquely on the vector v. We can identify v with this operator and thus generalize the concept of tangent vectors to differentiable manifolds M.

Definition 4.2.1 *A differentiable function $\gamma : (-\epsilon, \epsilon) \rightarrow M$ is called a (differentiable) curve in M.*

Let D be the set of all functions that are differentiable at point $p = \gamma(0) \in M$. The tangent vector to the curve γ at $t = 0$ is an operator (or function) $\gamma'(0) : D \rightarrow R$ given by

$$\gamma'(0) f = \frac{d}{dt}(f \circ \gamma) \mid_{t=0} .$$

A tangent vector at p is the tangent vector of some curve γ at $t = 0$ with $\gamma(0) = p$. The set of all tangent vectors at p is called the tangent space at p and denoted by $T_p M$.

Let $(x_1, \cdots, x_n)$ be the coordinates in a local chart (U, ϕ) that covers p with $\phi(p) = 0 \in \phi(U)$. Let $\gamma(t) = (x_1(t), \cdots, x_n(t))$ be a curve in M with $\gamma(0) = p$. Then

$$\begin{aligned}\gamma'(0) f &= \frac{d}{dt} f(\gamma(t)) \mid_{t=0} = \frac{d}{dt} f(x_1(t), \cdots, x_n(t)) \mid_{t=0} \\ &= \sum_i x_i'(0) \frac{\partial f}{\partial x_i} = \sum_i \left(x_i'(0) \left(\frac{\partial}{\partial x_i} \right)_p \right) f.\end{aligned}$$

This shows that the vector $\gamma'(0)$ can be expressed as the linear combination of $\left(\frac{\partial}{\partial x_i}\right)_p$, i.e.

$$\gamma'(0) = \sum_i x_i'(0) \left(\frac{\partial}{\partial x_i} \right)_p .$$

Here $\left(\frac{\partial}{\partial x_i}\right)_p$ is the tangent vector at p of the "coordinate curve":

$$x_i \rightarrow \phi^{-1}(0, \cdots, 0, x_i, 0, \cdots, 0);$$

and

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \frac{\partial}{\partial x_i} f(\phi^{-1}(0, \cdots, 0, x_i, 0, \cdots, 0)) \mid_{x_i=0} .$$

One can see that the tangent space T_pM is an n-dimensional linear space with the basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p .$$

The dual space of the tangent space T_pM is called the cotangent space, and denoted by T_p^*M. We can realize it in the following.

We first introduce the differential of a differentiable mapping.

Definition 4.2.2 *Let M and N be two differential manifolds and let $f : M \to N$ be a differentiable mapping. For every $p \in M$ and for each $v \in T_pM$, choose a differentiable curve $\gamma : (-\epsilon, \epsilon) \to M$, with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $\beta = f \circ \gamma$. Define the mapping*

$$df_p : T_pM \to T_{f(p)}N$$

by

$$df_p(v) = \beta'(0).$$

We call df_p the differential of f. Obviously, it is a linear mapping from one tangent space T_pM to the other $T_{f(p)}N$; And one can show that it does not depend on the choice of γ (See [Ca]). When $N = R^1$, the collection of all such differentials form the cotangent space. From the definition, one can derive that

$$df_p(\left(\frac{\partial}{\partial x_i}\right)_p) = \left(\frac{\partial}{\partial x_i}\right)_p f.$$

In particular, for the differential $(dx_i)_p$ of the coordinates function x_i, we have

$$(dx_i)_p(\left(\frac{\partial}{\partial x_j}\right)_p) = \delta_{ij}.$$

And consequently,

$$df_p = \sum_i \left(\frac{\partial f}{\partial x_i}\right)_p (dx_i)_p.$$

From here we can see that

$$(dx_1)_p, (dx_2)_p, \cdots, (dx_n)_p$$

form a basis of the cotangent space T_p^*M at point p.

Definition 4.2.3 *The tangent space of M is*

$$TM = \bigcup_{p \in M} T_pM.$$

And the cotangent space of M is

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

4.3 Riemannian Metrics

To measure the arc length, area, or volume on a manifold, we need to introduce some kind of measurements, or ‘metrics’. For a surface S in the three dimensional Euclidean space R^3, there is a natural way of measuring the length of vectors tangent to S, which are simply the length of the vectors in the ambient space R^3. These can be expressed in terms of the inner product $<,>$ in R^3. Given a curve $\gamma(t)$ on S for $t \in [a,b]$, its length is the integral

$$\int_a^b |\gamma'(t)| dt,$$

where the length of the velocity vector $\gamma'(t)$ is given by the inner product in R^3:

$$|\gamma'(t)| = \sqrt{< \gamma'(t), \gamma'(t) >}.$$

The definition of the inner product $<,>$ enable us to measure not only the lengths of curves on S, but also the area of domains on S, as well as other quantities in geometry.

Now we generalize this concept to differentiable manifolds without using the ambient space. More precisely, we have

Definition 4.3.1 *A Riemannian metric on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $< \cdot, \cdot >_p$ on the tangent space T_pM, which varies differentiably, that is,*

$$g_{ij}(q) \equiv < \left(\frac{\partial}{\partial x_i}\right)_q , \left(\frac{\partial}{\partial x_j}\right)_q >_q$$

is a differentiable function of q for all q near p.

A differentiable manifold with a given Riemannian metric will be called a Riemannian manifold.

It is clear this definition does not depend on the choice of coordinate system. Also one can prove that (see [Ca])

Proposition 4.3.1 *A differentiable manifold M has a Riemannian metric.*

Now we show how a Riemannian metric can be used to measure the length of a curve on a manifold M.

Let I be an open interval in R^1, and let $\gamma : I \rightarrow M$ be a differentiable mapping (curve) on M. For each $t \in I$, $d\gamma$ is a linear mapping from T_tI to $T_{\gamma(t)}M$, and $\gamma'(t) \equiv \frac{d\gamma}{dt} \equiv d\gamma(\frac{d}{dt})$ is a tangent vector of $T_{\gamma(t)}M$. The restriction of the curve γ to a closed interval $[a,b] \subset I$ is called a segment. We define the length of the segment by

$$\int_a^b < \gamma'(t), \gamma'(t) >^{1/2} dt.$$

Here the arc length element is

$$ds =< \gamma'(t), \gamma'(t) >^{1/2} dt. \tag{4.1}$$

As we mentioned in the previous section, let $(x_1, \cdots, x_n)$ be the coordinates in a local chart (U, ϕ) that covers $p = \gamma(t) = (x_1(t), \cdots, x_n(t))$. Then

$$\gamma'(t) = \sum_i x_i'(t) \left(\frac{\partial}{\partial x_i} \right)_p .$$

Consequently,

$$\begin{aligned} ds^2 &= < \gamma'(t), \gamma'(t) > dt^2 \\ &= \sum_{i,j=1}^{n} < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_p x_i'(t)dt\, x_j'(t)dt \\ &= \sum_{i,j=1}^{n} g_{ij}(p) dx_i dx_j . \end{aligned}$$

This expresses the length element ds in terms of the metric g_{ij}.

To derive the formula for the volume of a region (an open connected subset) D in M in terms of metric g_{ij}, let's begin with the volume of the parallelepiped form by the tangent vectors. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. Let

$$\frac{\partial}{\partial x_i}(p) = \sum_j a_{ij} e_j, \quad i = 1, \cdots n.$$

Then

$$g_{ik}(p) =< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} >_p = \sum_{j\,l} a_{ij} a_{kl} < e_j, e_l >= \sum_j a_{ij} a_{kj}. \tag{4.2}$$

We know the volume of the parallelepiped form by $\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_n}$ is the determinant $det(a_{ij})$, and by virtue of (4.2), it is the same as $\sqrt{det(g_{ij})}$.

Assume that the region D is cover by a coordinate chart (U, ϕ) with coordinates $(x_1, \cdots x_n)$. From the above arguments, it is natural to define the volume of D as

$$vol(D) = \int_{\phi(U)} \sqrt{det(g_{ij})} dx_1 \cdots dx_n .$$

A trivial example is $M = R^n$, the n-dimensional Euclidean space with

$$\frac{\partial}{\partial x_i} = e_i = (0, \cdots, 1, 0, \cdots, 0).$$

The metric is given by

$$g_{ij} =< e_i, e_j >= \delta_{ij}.$$

A less trivial example is S^2, the 2-dimensional unit sphere with standard metric, i.e., with the metric inherited from the ambient space $R^3 = \{(x, y, z)\}$. We will use the usual spherical coordinates (θ, ψ) with

$$\begin{cases} x = \sin\theta \cos\psi \\ y = \sin\theta \sin\psi \\ z = \cos\theta \end{cases}$$

to illustrate how we use the measurement (inner product) in R^3 to derive the expression of the standard metric g_{ij} in term of the local coordinate (θ, ψ) on S^2.

Let p be a point on S^2 covered by a local coordinates chart (U, ϕ) with coordinates (θ, ψ). We first derive the expression for $\left(\frac{\partial}{\partial\theta}\right)_p$ and $\left(\frac{\partial}{\partial\psi}\right)_p$ the tangent vectors at point $p = \phi^{-1}(\theta, \psi)$ of the coordinate curves $\psi =$ constant and $\theta =$constant respectively.

In the coordinates of R^3, we have,

$$\left(\frac{\partial}{\partial\theta}\right)_p = (\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}) = (\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta),$$

and

$$\left(\frac{\partial}{\partial\psi}\right)_p = (\frac{\partial x}{\partial\psi}, \frac{\partial y}{\partial\psi}, \frac{\partial z}{\partial\psi}) = (-\sin\theta\sin\psi, \sin\theta\cos\psi, 0).$$

It follows that

$$g_{11}(p) =< \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} >_p = \cos^2\theta\cos^2\psi + \cos^2\theta\sin^2\psi + \sin^2\theta = 1,$$

$$g_{12}(p) = g_{21}(p) =< \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\psi} >_p = 0,$$

and

$$g_{22}(p) =< \frac{\partial}{\partial\psi}, \frac{\partial}{\partial\psi} >_p = \sin^2\theta.$$

Consequently, the length element is

$$ds = \sqrt{g_{11}d\theta^2 + 2g_{12}d\theta d\psi + g_{22}d\psi^2} = \sqrt{d\theta^2 + \sin^2 d\psi^2},$$

and the area element is

$$dA = \sqrt{det(g_{ij})}d\theta d\psi = \sin\theta d\theta d\psi.$$

These conform with our knowledge on the length and area in the spherical coordinates.

4.4 Curvature

4.4.1 Curvature of Plane Curves

Consider curves on a plane. Intuitively, we feel that a straight line does not bend at all, and a circle of radius one bends more than a circle of radius 2. To measure the extent of bending of a curve, or the amount it deviates from being straight, we introduce curvature.

Let p be a point on a curve, and $\mathbf{T}(p)$ be the unit tangent vector at p. Let q be a nearby point. Let $\Delta\alpha$ be the amount of change of the angle from $\mathbf{T}(p)$ to $\mathbf{T}(q)$, and Δs the arc length between p and q on the curve. We define the curvature at p as

$$k(p) = \lim_{q \longrightarrow p} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}(p).$$

If a plane curve is given parametrically as $c(t) = (x(t), y(t))$, then the unit tangent vector at $c(t)$ is

$$\mathbf{T}(t) = \frac{(x'(t), y'(t))}{\sqrt{[x'(t)]^2 + [y'(t)]^2}},$$

and

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

By a straight forward calculation, one can verify that

$$k(c(t)) = \frac{|d\mathbf{T}|}{ds} = \left| \frac{x'(t)y''(t) - y'(t)x''(t)}{\{[x'(t)]^2 + [y'(t)]^2\}^{3/2}} \right|.$$

If a plane curve is given explicitly as $y = f(x)$, then in the above formula, replacing t by x and $y(t)$ by $f(x)$, we find that the curvature at point (x, f(x)) is

$$k(x, f(x)) = \left| \frac{f''(x)}{\{1 + [f'(x)]^2\}^{3/2}} \right|. \tag{4.3}$$

4.4.2 Curvature of Surfaces in R^3

1. Principle Curvature

Let S be a surface in R^3, p be a point on S, and $\mathbf{N}(p)$ a normal vector of S at p. Given a tangent vector $\mathbf{T}(p)$ at p, consider the intersection of the surface with the plane containing $\mathbf{T}(p)$ and $\mathbf{N}(p)$. This intersection is a plane curve, and its curvature is taken as the absolute value of the *normal curvature*, which is positive if the curve turns in the same direction as the surface's chosen normal, and negative otherwise. The normal curvature varies as the tangent vector $\mathbf{T}(p)$ changes. The maximum and minimum values of the normal curvature at point p are called *principal curvatures*, and usually denoted by $k_1(p)$ and $k_2(p)$.

2. Gaussian Curvature. One way to measure the extent of bending of a surface is by *Gaussian curvature.* It is defined as the product of the two principal curvatures

$$K(p) = k_1(p) \cdot k_2(p).$$

From this, one can see that no matter which side normal is chosen (say, for a closed surface, one may chose outer normals or inner normals), the Gaussian curvature of a sphere is positive, of a one sheet hyperboloids is negative, and of a torus or a plane is zero. A sphere of radius R has Gaussian curvature $\frac{1}{R^2}$.

Let $f(x, y)$ be a differentiable function. Consider a surface S in R^3 defined as the graph of $x_3 = f(x_1, x_2)$. Assume that the origin is on S, and the x-y plane is tangent to S, that is

$$f(0,0) = 0 \quad \text{and} \quad f_1(0,0) = 0 = f_2(0,0),$$

where, for simplicity, we write $f_i = \frac{\partial f}{\partial x_i}$ and we will use f_{ij} to denote $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

We calculate the Gaussian curvature of S at the origin $\mathbf{0} = (0,\ 0,\ 0)$.

Let $\mathbf{u} = (u_1, u_2)$ be a unit vector on the tangent plane. Then by (4.3), the normal curvature $k_n(\mathbf{u})$ in $\mathbf{u}$ direction is

$$\frac{\frac{\partial^2 f}{\partial \mathbf{u}^2}}{[(\frac{\partial f}{\partial \mathbf{u}})^2 + 1]^{3/2}}(\mathbf{0})$$

where $\frac{\partial f}{\partial \mathbf{u}}$ and $\frac{\partial^2 f}{\partial \mathbf{u}^2}$ are first and second directional derivative of f in $\mathbf{u}$ direction. Using the assumption that $f_1(\mathbf{0}) = 0 = f_2(\mathbf{0})$, we arrive at

$$k_n(\mathbf{u}) = (f_{11}u_1^2 + 2f_{12}u_1u_2 + f_{22}u_2^2)(\mathbf{0}) = \mathbf{u}^T \mathbf{F}\ \mathbf{u},$$

where

$$\mathbf{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} (\mathbf{0}),$$

and $\mathbf{u}^T$ denotes the transpose of $\mathbf{u}$.

Let λ_1 and λ_2 be two eigenvalues of the matrix $\mathbf{F}$ with corresponding unit eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, i.e.

$$\mathbf{F}\ \mathbf{e}_i = \lambda_i\, \mathbf{e}_i, \quad i = 1, 2. \tag{4.4}$$

Since the matrix $\mathbf{F}$ is symmetric, the two eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal, and it is obvious that

$$\mathbf{e}_i^T \mathbf{F}\ \mathbf{e}_i = \lambda_i, \quad i = 1, 2.$$

Let θ be the angle between $\mathbf{u}$ and $\mathbf{e}_1$. Then we can express

$$\mathbf{u} = \cos\theta\ \mathbf{e}_1 + \sin\theta\ \mathbf{e}_2.$$

It follows that

$$k_n(\mathbf{u}) = \mathbf{u}^T \mathbf{F}\ \mathbf{u} = \cos^2\theta\ \lambda_1 + \sin^2\theta\ \lambda_2.$$

Assume that $\lambda_1 \leq \lambda_2$. Then

$$\lambda_1 \leq \lambda_1 + \sin^2\theta\ (\lambda_2 - \lambda_1) = k_n(\mathbf{u}) = \cos^2\theta\ (\lambda_1 - \lambda_2) + \lambda_2 \leq \lambda_2.$$

Hence λ_1 and λ_2 are minimum and maximum values of normal curvature, and therefore they are principal curvatures k_1 and k_2.

From (4.4), we see that λ_1 and λ_2 are solutions of the quadratic equation

$$\begin{vmatrix} f_{11} - \lambda & f_{12} \\ f_{21} & f_{22} - \lambda \end{vmatrix} = \lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0.$$

Therefore, by definition, the Gaussian curvature at $\mathbf{0}$ is

$$K(\mathbf{0}) = k_1 k_2 = \lambda_1 \lambda_2 = (f_{11}f_{22} - f_{12}f_{21})(\mathbf{0}).$$

4.4.3 Curvature on Riemannian Manifolds

On a general n dimensional Riemannian manifold M, there are several kinds of curvatures. Before introducing them, we need a few preparations.

1. *Vector Fields and Brackets*

A vector field X on M is a correspondence that assigns each point $p \in M$ a vector $X(p)$ in the tangent space T_pM. It is a mapping from M to TM. If this mapping is differentiable, then we say the vector field X is differentiable. In a local coordinates, we can express

$$X(p) = \sum_i^n a_i(p) \frac{\partial}{\partial x_i}.$$

When applying to a smooth function f on M, it is a directional derivative operator in the direction $X(p)$:

$$(Xf)(p) = \sum_i^n a_i(p) \frac{\partial f}{\partial x_i}.$$

Definition 4.4.1 *For two differentiable vector fields X and Y, define the Lie Bracket as*

$$[X, Y] = XY - YX.$$

If $X(p) = \sum_i^n a_i(p) \frac{\partial}{\partial x_i}$ and $Y(p) = \sum_j^n b_j(p) \frac{\partial}{\partial x_j}$, then

$$[X, Y]f = XYf - YXf = \sum_{i,j=1}^n \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}.$$

2. *Covariant Derivatives and Riemannian Connections*

To be more intuitive, we begin with a surface S in R^3. Let $c: I \to S$ be a curve on S, and let $V(t)$ be a vector field along $c(t), t \in I$. One can see that $\frac{dV}{dt}$ is a vector in the ambient space R^3, but it may not belong to the tangent plane of S. To consider that rate of change of $V(t)$ restricted to the surface S, we make an orthogonal projection of $\frac{dV}{dt}$ onto the tangent plane, and denote it by $\frac{DV}{dt}$. We call this the covariant derivative of $V(t)$ on S – the derivative viewed by the two-dimensional creatures on S.

On an n-dimensional differentiable manifold M, the covariant derivative is defined by affine connections.

Let $D(M)$ be the set of all smooth vector fields.

Definition 4.4.2 *Let $X, Y, Z \in D(M)$, and let f and g be smooth functions on M. An affine connection D on a differentiable manifold M is a mapping from $D(M) \times D(M)$ to $D(M)$ which maps (X, Y) to $D_X Y$ and satisfies*

$$
\begin{aligned}
&1) D_{fX+gY} Z = f D_X Z + g D_Y Z \\
&2) D_X (Y + Z) = D_X Y + D_X Z \\
&3) D_X (fY) = f D_X Y + X(f) Y.
\end{aligned}
$$

Given an affine connection D on M, for a vector field $V(t) = Y(c(t))$ along a differentiable curve $c: I \to M$, define the covariant derivative of V along $c(t)$ as

$$\frac{DV}{dt} = D_{\frac{dc}{dt}} Y.$$

In a local coordinates, let

$$c(t) = (x_1(t), \cdots x_n(t)) \quad \text{and } V(t) = \sum_i^n v_i(t) \frac{\partial}{\partial x_i}.$$

Then by the properties of the affine connection, one can express

$$\frac{DV}{dt} = \sum_i \frac{dv_i}{dt} \frac{\partial}{\partial x_i} + \sum_{i,j} \frac{dx_i}{dt} v_j D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}. \tag{4.5}$$

Recall that in Euclidean spaces with inner product $<, >$, a vector field $V(t)$ along a curve $c(t)$ is parallel if only if $\frac{dV}{dt} = 0$; and for a pair of vector fields X and Y along $c(t)$, we have $< X, Y >=$ constant. Similarly, we have

Definition 4.4.3 *Let M be a differentiable manifold with an affine connection D. A vector field $V(t)$ along a curve $c(t) \in M$ is parallel if $\frac{DV}{dt} = 0$.*

Definition 4.4.4 *Let M be a differentiable manifold with an affine connection D and a Riemannian metric $<,>$. If for any pair of parallel vector fields X and Y along any smooth curve $c(t)$, we have*

$$< X, Y >_{c(t)} = \text{ constant },$$

then we say that D is compatible with the metric $<,>$.

The following fundamental theorem of Levi and Civita guarantees the existence of such an affine connection on a Riemannian manifold.

Theorem 4.4.1 *Given a Riemannian manifold M with metric $<,>$, there exists a unique connection D on M that is compatible with $<,>$. Moreover this connection is symmetric, i.e.*

$$D_X Y - D_Y X = [X, Y].$$

We skip the proof of the theorem. Interested readers may see the book of do Carmo [ca] (page 55).

In a local coordinates $(x_1, \cdots, x_n)$, this unique Riemannian connection can be expressed as

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

where

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m \left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right) g^{mk}$$

is called the Christoffel symbols, $g_{ij} =< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >$, and (g^{ij}) is the inverse matrix of (g_{ij}).

3. Curvature

Definition 4.4.5 *Let M be a Riemannian manifold with the Riemannian connection D. The* curvature *R is a correspondence that assigns every pair of smooth vector field $X, Y \in D(M)$ a mapping $R(X, Y) : D(M) \rightarrow D(M)$ defined by*

$$R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z, \quad \forall Z \in D(M).$$

Definition 4.4.6 *The sectional curvature of a given two dimensional subspace $E \subset T_p M$ at point p is*

$$K(E) = \frac{< R(X, Y)X, Y >}{|X|^2 |Y|^2 - < X, Y >^2},$$

where X and Y are any two linearly independent vectors in E, and

$$\sqrt{|X|^2 |Y|^2 - < X, Y >^2}$$

is the area of the parallelogram spun by X and Y.

One can show that $K(E)$ so defined is independent of the choice of X and Y. It is actually the Gaussian curvature of the section.

Example. Let M be the unit sphere with standard metric, then one can verify that its sectional curvature at every point is $K(E) = 1$.

Definition 4.4.7 *Let $X = Y_n$ be a unit vector in T_pM, and let $\{Y_1, \cdots, Y_{n-1}\}$ be an orthonormal basis of the hyper plane in T_pM orthogonal to X, then the Ricci curvature in the direction X is the average of the sectional curvature of the $n-1$ sections spun by X and Y_1, X and Y_2, $\cdots$, and X and Y_{n-1}:*

$$Ric_p(X) = \frac{1}{n-1} \sum_{i=1}^{n-1} < R(X, Y_i)X, Y_i > .$$

The scalar curvature at p is

$$R(p) = \frac{1}{n} \sum_{i=1}^{n} Ric_p(Y_i).$$

One can show that the above definition does not depend on the choice of the orthonormal basis. On two dimensional manifolds, Ricci curvature is the same as sectional curvature, and they both depends only on the point p.

4.5 Calculus on Manifolds

In the previous chapter, we introduced the concept of affine connection D on a differentiable manifold M. It is a mapping from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$, where $\Gamma(M)$ is the set of all smooth vector fields on M. Here we will extend this definition to the space of differentiable tensor fields, and thus introduce higher order covariant derivatives. For convenience, we adapt the summation convention and write, for instance,

$$X^i \frac{\partial}{\partial x_i} := \sum_i X^i \frac{\partial}{\partial x_i},$$

$$\Gamma^j_{ik} dx^k := \sum_k \Gamma^j_{ik} dx^k,$$

and so on.

4.5.1 Higher Order Covariant Derivatives and the Laplace-Bertrami Operator

Let p and q be two non-negative integers. At a point x on M, as usual, we denote the tangent space by T_xM. Define $T^q_p(T_xM)$ as the space of (p, q)-tensors on T_xM, that is, the space of $(p + q)$-linear forms

$$\eta : \underbrace{T_xM \times \cdots \times T_xM}_{p} \times \underbrace{T_x^*M \times \cdots \times T_x^*M}_{q} \to R^1.$$

In a local coordinates, we can express

$$T = T_{i_1 \cdots i_p}^{j_1 \cdots j_q} dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_q}}.$$

Recall that, for the affine connection D, in a local coordinates, if we set

$$\nabla_i = D_{\frac{\partial}{\partial x_i}} ,$$

then

$$\nabla_i \left(\frac{\partial}{\partial x_j} \right)(x) = \Gamma_{ij}^k \left(\frac{\partial}{\partial x_k} \right)_x ,$$

where Γ_{ij}^k are the Christoffel symbols. For smooth vector fields $X, Y \in \Gamma(M)$ with

$$X = (X^1, \cdots, X^n) \quad \text{and} \quad Y = (Y^1, \cdots, Y^n),$$

by the bilinear properties of the connection, we have

$$\begin{aligned} D_X Y &= D_{X^i \frac{\partial}{\partial x_i}} Y = X^i \nabla_i Y \\ &= X^i \left(\frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x_j} + Y^j \Gamma_{ij}^k \frac{\partial}{\partial x_k} \right) \\ &= X^i \left(\frac{\partial Y^j}{\partial x_i} + \Gamma_{ik}^j Y^k \right) \frac{\partial}{\partial x_j} \end{aligned} \tag{4.6}$$

Now we naturally extend this affine connection D to differentiable (p,q) tensor fields T on M. For a point x on M, and a vector $X \in T_xM$, $D_X T$ is defined to be a (p,q)-tensor on T_xM by

$$D_X T(x) = X^i (\nabla_i T)(x),$$

where

$$\begin{aligned} (\nabla_i T)(x)_{i_1 \cdots i_p}^{j_1 \cdots j_q} = &\left(\frac{\partial T_{i_1 \cdots i_p}^{j_1 \cdots j_q}}{\partial x_i} \right)_x - \sum_{k=1}^p \Gamma_{i i_k}^{\alpha}(x) T(x)_{i_1 \cdots i_{k-1} \alpha i_{k+1} \cdots i_p}^{j_1 \cdots j_q} \\ &+ \sum_{k=1}^q \Gamma_{i\alpha}^{j_k}(x) T(x)_{i_1 \cdots i_p}^{j_1 \cdots j_{k-1} \alpha j_{k+1} j_q}. \end{aligned} \tag{4.7}$$

Notice that (4.6) is a particular case when (4.7) is applied to the $(0,1)$-tensor Y. If we apply it to a $(1,0)$-tensor, say dx^j, then

$$\nabla_i (dx^j) = -\Gamma_{ik}^j dx^k. \tag{4.8}$$

Given two differentiable tensor fields T and $\tilde{T}$, we have

$$D_X(T \otimes \tilde{T}) = D_X T \otimes \tilde{T} + T \otimes D_X \tilde{T}.$$

For a differentiable (p, q)-tensor field T, we define $\bigtriangledown T$ to be the $(p+1, q)$-tensor field whose components are given by

$$(\bigtriangledown T)^{j_1 \cdots j_q}_{i_1 \cdots i_{p+1}} = (\bigtriangledown_{i_1} T)^{j_1 \cdots j_q}_{i_2 \cdots i_{p+1}}.$$

Similarly, one can define $\bigtriangledown^2 T$, $\bigtriangledown^3 T$, $\cdots$.

For example, a smooth function $f(x)$ on M is a $(0,0)$-tensor. Hence $\bigtriangledown f$ is a $(1,0)$-tensor defined by

$$\bigtriangledown f = \bigtriangledown_i f dx^i = \frac{\partial f}{\partial x_i} dx^i.$$

And $\bigtriangledown^2 f$ is a $(2,0)$-tensor:

$$\begin{aligned}
\bigtriangledown^2 f &= \bigtriangledown_j \left(\frac{\partial f}{\partial x_i} dx^i \right) \otimes dx^j \\
&= \left(\bigtriangledown_j \left(\frac{\partial f}{\partial x_i} \right) dx^i + \frac{\partial f}{\partial x_i} \bigtriangledown_j (dx^i) \right) dx^j \\
&= \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \right) dx^i \otimes dx^j.
\end{aligned}$$

Here we have used (4.7) and (4.8). In the Riemannian context, $\bigtriangledown^2 f$ is called the Hessian of f and denoted by Hess(f). Its ij^{th} component is

$$(\bigtriangledown^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k}.$$

The trace of the Hessian matrix $((\bigtriangledown^2 f)_{ij})$ is defined to be the Laplace-Bertrami operator $\triangle$ acting on f,

$$\triangle f = g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \right)$$

where (g^{ij}) is the inverse matrix of (g_{ij}). Through a straight forward calculation, one can verify that

$$\triangle = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial x_j})$$

where $|g|$ is the determinant of (g_{ij}).

4.5.2 Integrals

On a smooth n-dimensional Riemannian manifold (M, g), one can define a natural positive Radon measure on M, and the theory of Lebesgue integral applies.

Let $(U_i, \phi_i)_i$ be a family of coordinates charts that covers M. We say that a family $(U_i, \phi_i, \eta_i)_i$ is a partition of unity associate to $(U_i, \phi_i)_i$ if

(i) $(\eta_i)_i$ is a smooth partition of unity associated to the covering $(U_i)_i$, and

(ii) for any i, supp $\eta_i \subset U_i$.

One can show that, for each family of coordinates chart $(U_i, \phi_i)_i$ of M, there exists a corresponding family of partition of unity $(U_i, \phi_i, \eta_i)_i$.

Let f be a continuous function on M with compact support, and given a family of coordinates charts $(U_i, \phi_i)_i$ of M, we define its integral on M as the sum of the integrals on open sets in R^n:

$$\int_M f dV_g = \sum_i \int_{\phi_i(U_i)} (\eta_i \sqrt{|g|} f) \circ \phi_i^{-1} dx$$

where $(U_i, \phi_i, \eta_i)_i$ is a family of partition of unity associated to $(U_i, \phi_i)_i$, $|g|$ is the determinant of (g_{ij}) in the charts $(U_i, \phi_i)_i$, and dx stands for the volume element of R^n. One can prove that such a definition does not depend on the choice of the charts $(U_i, \phi_i)_i$ and the partition of unity $(U_i, \phi_i, \eta_i)_i$.

For smooth functions u and v on a compact manifold M, one can verify the following integration by parts formula

$$-\int_M \triangle_g u \, v dV_g = \int_M < \triangledown u, \triangledown v > dV_g, \tag{4.9}$$

where $\triangle_g$ is the Laplace-Bertrami operator associated to the metric g, and

$$< \triangledown u, \triangledown v >= g^{ij} \triangledown_i u \triangledown_j v = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}$$

is the scalar product associated with g for 1-forms.

4.5.3 Equations on Prescribing Gaussian and Scalar Curvature

A Riemannian metric g on M is said to be point-wise conformal to another metric g_o if there exist a positive function ρ, such that

$$g(x) = \rho(x) g_o(x), \;\; \forall \, x \in M.$$

Suppose M is a two dimensional Riemannian manifold with a metric g_o and the corresponding Gaussian curvature $K_o(x)$. Let $g(x) = e^{2u(x)} g_o(x)$ for some smooth function $u(x)$ on M. Let $K(x)$ be the Gaussian curvature associated to the metric g. Then by a straight forward calculation, one can verify that

$$-\triangle_o u + K_o(x) = K(x) e^{2u(x)}, \;\; \forall \, x \in M. \tag{4.10}$$

Here $\triangle_o$ is the Laplace-Bertrami operator associated to g_o.

Similarly, for conformal relation between scalar curvatures on n-dimensional manifolds ($n \geq 3$), say on standard sphere S^n, we have

$$-\triangle_o u + \frac{n(n-2)}{4} u = \frac{(n-2)}{4(n-1)} R(x) u^{\frac{n+2}{n-2}}, \;\; \forall\, x \in S^n, \tag{4.11}$$

where g_o is the standard metric on S^n with the corresponding Laplace-Bertrami operator $\triangle_o$, $R(x)$ is the scalar curvature associated to the point-wise conformal metric $g = u^{\frac{4}{n-2}} g_o$.

In Chapter 5 and 6, we will study the existence and qualitative properties of the solutions u for the above two equations respectively.

4.6 Sobolev Embeddings

Let (M, g) be a smooth Riemannian manifold. Let u be a smooth function on M and k be a non-negative integer. We denote $\bigtriangledown^k u$ the k^{th} covariant derivative of u as defined in the previous section, and its norm $|\bigtriangledown^k u|$ is defined in a local coordinates chart by

$$|\bigtriangledown^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\bigtriangledown^k u)_{i_1 \cdots i_k} (\bigtriangledown^k u)_{j_1 \cdots j_k}.$$

In particular, we have $|\bigtriangledown^0 u| = |u|$ and

$$|\bigtriangledown^1 u|^2 = |\bigtriangledown u|^2 = g^{ij} (\bigtriangledown u)_i (\bigtriangledown u)_j = g^{ij} \bigtriangledown_i u \bigtriangledown_j u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

For an integer k and a real number $p \geq 1$, let $\mathcal{C}_k^p(M)$ be the space of C^∞ functions on M such that

$$\int_M |\bigtriangledown^j u|^p dV_g < \infty \,, \forall\, j = 0, 1, \cdots, k.$$

Definition 4.6.1 *The Sobolev space $H^{k,p}(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect to the norm*

$$\|u\|_{H^{k,p}(M)} = \sum_{j=0}^{k} \left(\int_M |\bigtriangledown^j u|^p dV_g \right)^{1/p}.$$

Many results concerning the Sobolev spaces on R^n introduced in Chapter 1 can be generalized to Sobolev spaces on Riemannian manifolds. For application purpose, we list some of them in the following. Interested readers may find the proofs in [He] or [Au].

Theorem 4.6.1 *For any integer k, $H^{k,2}(M)$ is a Hilbert space when equipped with the equivalent norm*

$$\|u\| = \left(\sum_{i=1}^{k} \int_M |\nabla^i u|^2 dV_g \right)^{1/2} .$$

The corresponding scalar product is defined by

$$< u, v >= \sum_{i=1}^{k} \int_M < \nabla^i u, \nabla^i v > dV_g$$

where $< \cdot, \cdot >$ is the scalar product on covariant tensor fields associated to the metric g.

Theorem 4.6.2 *If M is compact, then $H^{k,p}(M)$ does not depend on the metric.*

Theorem 4.6.3 *(*Sobolev Embeddings I*)*

Let (M, g) be a smooth, compact n-dimensional Riemannian manifold. Assume $1 \leq q < n$ and $p = \frac{nq}{n-q}$. Then

$$H^{1,q}(M) \subset L^p(M) .$$

More generally, if m, k are two integers with $0 \leq m < k$, and if $p = \frac{nq}{n-q(k-m)}$, then

$$H^{k,q}(M) \subset H^{m,p}(M) .$$

Theorem 4.6.4 *(*Sobolev Embeddings II*)*

Let (M, g) be a smooth, compact n-dimensional Riemannian manifold. Assume that $q \geq 1$ and m, k are two integers with $0 \leq m < k$. If $q > \frac{n}{k-m}$, then

$$H^{k,q}(M) \subset C^m(M) .$$

Theorem 4.6.5 *(*Compact Embeddings*)*

Let (M, g) be a smooth, compact n-dimensional Riemannian manifold.

(i) Assume that $q \geq 1$ and m, k are two integers with $0 \leq m < k$. Then for any real number p such that $1 \leq p < \frac{nq}{n-q(k-m)}$, the embedding

$$H^{k,q}(M) \subset H^{m,p}(M)$$

is compact. In particular, the embedding of $H^{1,q}(M)$ into $L^p(M)$ is compact if $1 \leq p < \frac{nq}{n-q}$.

(ii) For $q > n$, the embedding of $H^{k,q}(M)$ into $C^\lambda(M)$ is compact, if $(k - \lambda) > \frac{n}{q}$. In particular, the embedding of $H^{1,q}(M)$ into $C^0(M)$ is compact.

Theorem 4.6.6 *(*Poincaré Inequality*)*

Let (M, g) be a smooth, compact n-dimensional Riemannian manifold and let $p \in [1, n)$ be real. Then there exists a positive constant C such that for any $u \in H^{1,p}(M)$,

$$\left(\int_M |u - \bar{u}|^p dV_g \right)^{1/p} \leq C \left(\int_M |\nabla u|^p dV_g \right)^{1/p},$$

where

$$\bar{u} = \frac{1}{Vol_{(M,g)}} \int_M u dV_g$$

is the average of u on M.

5

Prescribing Gaussian Curvature on Compact 2-Manifolds

In this and the next chapters, we will present the readers with real research examples–semi-linear equations arising from prescribing Gaussian and scalar curvatures. We will illustrate, among other methods, how the calculus of variations can be applied to seek weak solutions of these equations.

To show the existence of weak solutions for a certain equation, we consider the corresponding functional $J(\cdot)$ in an appropriate Hilbert space. According to the compactness of the functional, people usually divide it into three cases:

a) subcritical case where the level set of J is compact, say, J satisfies the (PS) condition (see also Chapter 2):

Every sequence $\{u_k\}$ *that satisfies* $J(u_k){\rightarrow}c$ *and* $J'(u_k){\rightarrow}0$ *possesses a convergent subsequence,*

b) critical case which is the limiting situation, and can be approximated by subcritical cases, or

c) super critical case–the remaining cases.

To roughly see when a functional may lose compactness, let's consider $J(x) = (x^2-1)e^x$ defined on one dimensional space R^1. One can easily see that there exists a sequence $\{x_k\}$, for instance, $x_k = -k$, such that $J(x_k){\rightarrow}0$ and $J'(x_k){\rightarrow}0$ as $k{\rightarrow}\infty$, however, $\{x_k\}$ possesses no convergent subsequence because this sequence is not bounded. The functional has no compactness at level

$J = 0$. We know that in a finite dimensional space, a bounded sequence has a convergent subsequence. While in infinite dimensional space, even a bounded sequence may not have a convergent subsequence. For instance $\{\sin kx\}$ is a bounded sequence in $L^2([0,1])$ but does not have a convergent subsequence.

In the situation where the lack of compactness occurs, to find a critical point of the functional, one would try to recover the compactness through various means. Take again the one dimensional example $J(x) = (x^2 - 1)e^x$, which has no compactness at level $J = 0$. However, if we restrict it to levels less than 0, we can recover the compactness. That is, if we pick a sequence $\{x_k\}$ satisfying $J(x_k) < 0$ and $J'(x_k) \to 0$, then one can verify that it possesses a subsequence which converges to a critical point x_0 of J. Actually, this x_0 is a minimum of J.

To further illustrate the concept of compactness, we consider some examples in infinite dimensional space. Let Ω be an open bounded subset in R^n, and let $H_0^1(\Omega)$ be the Hilbert space introduced in previous chapters. Consider the Dirichlet boundary value problem

$$\begin{cases} -\triangle u = u^p(x)\,, & x \in \Omega \\ u(x) = 0\,, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

To seek weak solutions, we study the corresponding functional

$$J_p(u) = \int_\Omega |u|^{p+1} dx$$

on the constraint

$$H = \{u \in H_0^1(\Omega) \mid \int_\Omega |\bigtriangledown u|^2 dx = 1\}.$$

When $p < \frac{n+2}{n-2}$, it is in the subcritical case. As we have seen in Chapter 2, the functional satisfies the (PS) condition. If we define

$$m_p = \inf_{u \in H} J_p(u),$$

then a minimizing sequence $\{u_k\}$, $J_p(u_k) \to m_p$ possesses a convergent subsequence in $H_0^1(\Omega)$, and the limiting function u_0 would be the minimum of J_p in H, hence it is the desired weak solution.

When $p = \tau := \frac{n+2}{n-2}$, it is in the critical case. One can find a minimizing sequence that does not converge. For example, when $\Omega = B_1(0)$ is a ball, consider

$$u_k(x) = u(x) = \frac{[n(n-2)k]^{\frac{n-2}{4}}}{(1 + k|x|^2)^{\frac{n-2}{2}}} \eta(x)$$

where $\eta \in C_0^\infty(\Omega)$ is a cut off function satisfying

$$\eta(x) = \begin{cases} 1\,, & x \in B_{\frac{1}{2}}(0) \\ \text{between 0 and 1}\;, & \text{elsewhere}\,. \end{cases}$$

Let

$$v_k(x) = \frac{u_k(x)}{\int_\Omega |\bigtriangledown u_k|^2 dx}.$$

Then one can verify that $\{v_k\}$ is a minimizing sequence of J_τ in H with $J'(v_k) \to 0$. However it does not have a convergent subsequence. Actually, as $k \to \infty$, the energy of $\{v_k\}$ are concentrating at one point 0. We say that the sequence "blows up " at point 0.

In essence, the compactness of the functional J_τ here relies on the Sobolev embedding:

$$H^1(\Omega) \hookrightarrow L^{p+1}(\Omega).$$

As we learned in Chapter 1, this embedding is compact when $p < \frac{n+2}{n-2}$ and is not when $p = \frac{n+2}{n-2}$.

In the critical case, the compactness may be recovered by considering other level sets or by changing the topology of the domain. For example, if Ω is an annulus, then the above J_τ has a minimizer in H. In this and the next chapter, the readers will see examples from prescribing Gaussian and scalar curvature, where the corresponding equations are in critical case and where various means have been exploited to recover the compactness.

5.1 Prescribing Gaussian Curvature on S^2

Given a function $K(x)$ on two dimensional unit sphere $S := S^2$ with standard metric g_o, can it be realized as the Gaussian curvature of some point-wise conformal metric g? This has been an interesting problem in differential geometry and is known as the Nirenberg problem. If we let $g = e^{2u} g_o$, then it is equivalent to solving the following semi-linear elliptic equation on S^2:

$$-\triangle u + 1 = K(x) e^{2u}, \;\; x \in S^2. \tag{5.2}$$

where $\triangle$ is the Laplace-Bertrami operator associated with the standard metric g_o.

If we replace $2u$ by u and let $R(x) = 2K(x)$, then equation (5.2) becomes

$$-\triangle u + 2 = R(x) e^{u(x)}. \tag{5.3}$$

Here 2 and $R(x)$ are actually the scalar curvature of S^2 with standard metric g_o and with the point-wise conformal metric $e^u g_o$, respectively.

5.1.1 Obstructions

For equation (5.3) to have a solution, the function $R(x)$ must satisfy the obvious Gauss-Bonnet sign condition

$$\int_S R(x) e^u dx = 8\pi. \tag{5.4}$$

This can be seen by integrating both sides of the equation (5.3), and the fact that

$$-\int_S \triangle u \, dx = \int_S \nabla u \cdot \nabla 1 \, dx = 0.$$

Condition (5.4) implies that $R(x)$ must be positive somewhere. Besides this obvious necessary condition, there are other necessary conditions found by Kazdan and Warner. It reads

$$\int_S \nabla \phi_i \nabla R(x) e^u \, dx = 0, \quad i = 1, 2, 3. \tag{5.5}$$

Here ϕ_i are the first spherical harmonic functions, i.e

$$-\triangle \phi_i = 2\phi_i(x), \quad i = 1, 2, 3.$$

Actually, $\phi_i(x) = x_i$ in the coordinates $x = (x_1, x_2, x_3)$ of R^3.

Later, Bouguignon and Ezin generalized this condition to

$$\int_S X(R) e^u \, dx = 0. \tag{5.6}$$

where X is any conformal Killing vector field on standard S^2.

Condition (5.5) gives rise to many examples of $R(x)$ for which the equation (5.3) has no solution. In particular, a monotone rotationally symmetric function admits no solution. Then for which kinds of functions $R(x)$, can one solve (5.3)? This has been an interesting problem in geometry.

5.1.2 Variational Approach and Key Inequalities

To obtain the existence of a solution for equation (5.3), people usually use a variational approach. First prove the existence of a weak solution, then by a regularity argument, one can show that the weak solution is smooth and hence is the classical solution.

To obtain a weak solution, we let

$$H^1(S) := H^{1,2}(S)$$

be the Hilbert space with norm

$$\|u\|_1 = \left[\int_S (|\nabla u|^2 + u^2) dx \right]^{1/2}.$$

Let

$$H(S) = \{u \in H^1(S) \mid \int_S u dx = 0, \ \int_S R(x) e^u dx > 0\}.$$

Then by the Poincaré inequality, we can use

$$\|u\| = \left[\int_S |\bigtriangledown u|^2 \, dx\right]^{1/2}$$

as an equivalent norm in $H(S)$.

Consider the functional

$$J(u) = \frac{1}{2}\int_S |\bigtriangledown u|^2 \, dx - 8\pi \ln \int_S R(x)e^u \, dx$$

in $H(S)$.

To estimate the value of the functional J, we introduce some useful inequalities.

Lemma 5.1.1 *(Moser-Trudinger Inequality)*

Let S be a compact two dimensional Riemannian manifold. Then there exists constant C, such that the inequality

$$\int_S e^{4\pi u^2} dA \leq C \tag{5.7}$$

holds for all $u \in H^1(S)$, satisfying

$$\int_S |\bigtriangledown u|^2 dA \leq 1 \quad \text{and} \quad \int_S u dA = 0. \tag{5.8}$$

We skip the proof of this inequality. Interested readers many see the article of Moser [Mo1] or the article of Chen [C1].

From this inequality, one can derive immediately that

Lemma 5.1.2 *There exists constant C, such that for all $u \in H^1(S)$ holds*

$$\int_S e^u dA \leq C \exp\left\{\frac{1}{16\pi}\int_S |\bigtriangledown u|^2 dA + \frac{1}{4\pi}\int_S u dA\right\}. \tag{5.9}$$

Proof. Let

$$\bar{u} = \frac{1}{4\pi}\int_S u(x) dA$$

be the average of u on S. Still denote

$$\|u\| = \left[\int_S |\bigtriangledown u|^2 \, dx\right]^{1/2}.$$

i) For any $v \in H^1(S)$ with $\bar{v} = 0$, let $w = \frac{v}{\|v\|}$. Then $\|w\| = 1$ and $\bar{w} = 0$. Applying Lemma 5.1.1 to such w, we have

$$\int_S e^{\frac{4\pi v^2}{\|v\|^2}} dA \leq C. \tag{5.10}$$

From the obvious inequality

$$\left(\frac{\|v\|}{4\sqrt{\pi}} - \frac{2\sqrt{\pi}v}{\|v\|} \right)^2 \geq 0,$$

we deduce immediately

$$v \leq \frac{\|v\|^2}{16\pi} + \frac{4\pi v^2}{\|v\|^2}.$$

Together with (5.10), we have

$$\int_S e^v dA \leq C \exp\left\{ \frac{1}{16\pi} \|v\|^2 \right\} , \quad \forall v \in H^1(S), \text{ with } \bar{v} = 0. \tag{5.11}$$

ii) For any $u \in H^1(S)$, let $v = u - \bar{u}$. Then $\bar{v} = 0$, and we can apply (5.11) to v and obtain

$$\int_S e^{u-\bar{u}} dA \leq C \exp\left\{ \frac{1}{16\pi} \|u\|^2 \right\}.$$

Consequently

$$\int_S e^u dA \leq C \exp\left\{ \frac{1}{16\pi} \|u\|^2 + \bar{u} \right\}.$$

This completes the proof of the Lemma. □

From Lemma 5.1.2, we can conclude that the functional $J(\cdot)$ is bounded from below in $H(S)$. In fact, by the Lemma,

$$\int_S R(x) e^u dA \leq \max R \int_S e^u dA \leq \max R(x) \cdot C \exp\left\{ \frac{1}{16\pi} \|u\|^2 \right\}.$$

It follows that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - 8\pi \left[\ln(C \max R) + \frac{1}{16\pi} \|u\|^2 \right] \\ &\geq -8\pi \ln(C \max R) \end{aligned} \tag{5.12}$$

Therefore, the functional J is bounded from below in $H(S)$. Now we can take a minimizing sequence $\{u_k\} \subset H(S)$:

$$J(u_k) \to \inf_{u \in H(S)} J(u).$$

However, as we mentioned before, a minimizing sequence may not be bounded, that is, it may 'leak' to infinity. To prevent this from happening, we need some coerciveness of the functional. Although it is not true in general, we do have coerciveness for J in some special subset of $H(S)$, In particular, in the set of even functions–antipodal symmetric functions. More precisely, we have the following "Distribution of Mass Principle" from Chen and Li [CL7] [CL8].

Lemma 5.1.3 *Let Ω_1 and Ω_2 be two subsets of S such that*

$$dist(\Omega_1, \Omega_2) \geq \delta_o > 0.$$

Let $0 < \alpha_o \leq \frac{1}{2}$. Then for any $\epsilon > 0$, there exists a constant $C = C(\alpha_o, \delta_o, \epsilon)$, such that the inequality

$$\int_S e^u dA \leq C \exp\left\{\left(\frac{1}{32\pi} + \epsilon\right) \|u\|^2 + \bar{u}\right\} \tag{5.13}$$

holds for all $u \in H^1(S)$ satisfying

$$\frac{\int_{\Omega_1} e^u dA}{\int_S e^u dA} \geq \alpha_o \quad \textit{and} \quad \frac{\int_{\Omega_2} e^u dA}{\int_S e^u dA} \geq \alpha_o. \tag{5.14}$$

Intuitively, if we think $e^{u(x)}$ as the density of a solid sphere S at point x, then $\int_{\Omega_i} e^u dA$ is the mass of the region Ω_i, and $\int_S e^u dA$ the total mass of the sphere. The Lemma says that if the mass of the sphere is kind of distributed in two parts of the sphere, more precisely, if the total mass does not concentrate at one point, then we can have a stronger inequality than (5.9). In particular, for even functions, this stronger inequality (5.13) holds. If we apply this stronger inequality to the family of even functions in $H(S)$, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - 8\pi \left[\ln(C \max R) + \left(\frac{1}{32\pi} + \epsilon\right) \|u\|^2\right] \\ &\geq -8\pi \ln(C \max R) + \left(\frac{1}{4} - 8\pi\epsilon\right) \|u\|^2. \end{aligned}$$

Choosing ϵ small enough, we arrive at

$$J(u) \geq \frac{1}{8}\|u\|^2 - C_1, \quad \text{for all even functions } u \text{ in } H(S). \tag{5.15}$$

This stronger inequality guarantees the bounded-ness of a minimizing sequence, and hence it converges to a weak limit u_o in $H^1(S)$. As we will show in the next section, the functional J is weakly lower semi-continuous, and therefore, u_o so obtained is the minimum of $J(u)$, i.e., a weak solution of equation (5.2). Then by a standard regularity argument, as we will see in the next next section, we will be able to conclude that u_o is a classical solution.

In many articles, the authors use the "center of mass " analysis, i.e., consider the center of mass of the sphere with density $e^{u(x)}$ at point x:

$$P(u) = \frac{\int_S \mathbf{x}\, e^u dA}{\int_S e^u dA},$$

and use this to study the behavior of a minimizing sequence (See [CD], [CD1], [CY], and [CY1]). Both analysis are equivalent on the sphere S^2. However, since "the center of mass" $P(u)$ lies in the ambient space R^3, on a general two dimensional compact Riemannian surfaces, this kinds of analysis can not be applied. While the "Distribution of Mass" analysis can be applied to any compact surfaces, even surfaces with conical singularities. Interested reader may see the author's paper [CL7] [CL8]for more general version of inequality (5.13).

Proof of Lemma 5.1.3.
Let g_1, g_2 be two smooth functions, such that

$$1 \geq g_i \geq 0, \quad g_i(x) \equiv 1, \quad for\; x \in \Omega_i, \quad i = 1, 2$$

and supp $g_1 \cap$ supp $g_2 = \emptyset$. It suffice to show that for $u \in H^1(S)$ with $\int_S u dA = 0$, (5.14) implies

$$\int_S e^u dA \leq C \exp\left\{(\frac{1}{32\pi} + \epsilon)\|u\|^2\right\}. \tag{5.16}$$

Case i) If $\|g_1 u\| \leq \|g_2 u\|$, then by (5.9)

$$\begin{aligned}\int_S e^u dA &\leq \frac{1}{\alpha_o}\int_{\Omega_1} e^u dA \leq \frac{1}{\alpha_o}\int_S e^{g_1 u} dA \\ &\leq \frac{C}{\alpha_o}\exp\{\frac{1}{16\pi}\|g_1 u\|^2 + \overline{g_1 u}\} \\ &\leq \frac{C}{\alpha_o}\exp\{\frac{1}{32\pi}\|g_1 u + g_2 u\|^2 + \overline{g_1 u}\} \\ &\leq \frac{C}{\alpha_o}\exp\{\frac{1}{32\pi}(1+\epsilon_1)\|u\|^2 + c_1(\epsilon_1)\|u\|^2_{L^2}\}.\end{aligned} \tag{5.17}$$

for some small $\epsilon_1 > 0$. Here we have used the fact that

$$\begin{aligned}\|g_1 u + g_2 u\|^2 &= \int_S |\bigtriangledown(g_1 u + g_2 u)|^2 dA \\ &= \int_S |(\bigtriangledown g_1 + \bigtriangledown g_2)u + (g_1 + g_2)\bigtriangledown u|^2 dA \\ &\leq \|u\|^2 + C_1\|u\|\|u\|_{L^2} + C_2\|u\|^2_{L^2}.\end{aligned}$$

In order to get rid of the term $\|u\|^2_{L^2}$ on the right hand side of the above inequality, we employ the condition $\int_S u dA = 0$.

Given any $\eta > 0$, choose a, such that $meas\{x \in S | u(x) \geq a\} = \eta$. Apply (5.17) to the function $(u-a)^+ \equiv \max\{0, (u-a)\}$, we have

$$\begin{aligned}\int_S e^u dA &\leq e^a \int_S e^{u-a} dA \leq e^a \cdot \int_S e^{(u-a)^+} dA \\ &\leq C \exp\{\frac{1}{32\pi}(1+\epsilon_1)\|u\|^2 + c_1(\epsilon_1)\|(u-a)^+\|^2_2 + a\}\end{aligned} \tag{5.18}$$

Where $C = C(\delta, \alpha_o)$.

By the *Hölder* and Sobolev inequality (See Theorem 4.6.3)

$$\|(u-a)^+\|_{L^2}^2 \leq \eta^{\frac{1}{2}} \|(u-a)^+\|_{L^4}^2 \leq c\eta^{\frac{1}{2}} (\|u\|^2 + \|(u-a)^+\|_{L^2}^2) \tag{5.19}$$

Choose η so small that $C\eta^{1/2} \leq \frac{1}{2}$, then the above inequality implies

$$\|(u-a)^+\|_{L^2}^2 \leq 2C\eta^{\frac{1}{2}} \|u\|^2. \tag{5.20}$$

Now by Poincaré inequality (See Theorem 4.6.6)

$$a \cdot \eta \leq \int_{u \geq a} u dA \leq \int_S |u| dA \leq c\|u\|$$

hence for any $\delta > 0$,

$$a \leq \delta \|u\|^2 + \frac{c^2}{4\delta\eta^2}. \tag{5.21}$$

Now (5.16) follows from (5.18), (5.20) and (5.21).

Case ii) If $\|g_2 u\| \leq \|g_1 u\|$, by a similar argument as in case i), we obtain (5.17) and then (5.16). This completes the proof.

5.1.3 Existence of Weak Solutions

Theorem 5.1.1 *(Moser) If the function R is even, that is if*

$$R(x) = R(-x) \ \ \forall\, x \in S^2,$$

where x and $-x$ are two antipodal points. Then the necessary and sufficient condition for equation (5.3) to have a solution is $R(x)$ be positive somewhere.

To prove the existence of a weak solution, as in the previous section, we consider the functional

$$J(u) = \frac{1}{2} \int_S |\bigtriangledown u|^2 \, dx - 8\pi \ln \int_S R(x) e^u \, dx$$

in $H_e(S) = \{u \in H(S) \mid u \text{ is even almost everywhere }\}$. Let $\{u_k\}$ be a minimizing sequence of J in $H_e(S)$. Then by (5.15), $\{u_k\}$ is bounded in $H^1(S)$, and hence possesses a subsequence (still denoted by $\{u_k\}$) that converges weakly to some element u_o in $H^1(S)$. Then by the Compact Sobolev Embedding of H^1 into L^2

$$u_k {\rightarrow} u_o \quad \text{in } L^2(S). \tag{5.22}$$

Consequently

$$u_o(-x) = u_o(x) \ \text{ almost everywhere } \ , \text{ and } \int_S u_o(x) dA = 0,$$

that is $u_o \in H_e(S)$.

Furthermore, as we showed in the previous chapter, the integral is weakly lower semi-continuous, i.e.

$$\underline{\lim}_{k \to \infty} \int_S |\bigtriangledown u_k|^2 dA \geq \int_S |\bigtriangledown u_o|^2 dA.$$

To show that the functional J is weakly lower semi-continuous:

$$J(u_o) \leq \underline{\lim} J(u_k) = \inf_{u \in H_e(S)} J(u).$$

We need the following

Lemma 5.1.4 *If $\{u_k\}$ converges weakly to u in $H^1(S)$, then there exists a subsequence (still denoted by $\{u_k\}$), such that*

$$\int_S e^{u_k(x)} dA \to \int_S e^{u(x)} dA \,, \;\; \textit{as } k \to \infty.$$

We postpone the proof of this lemma for a moment. Now from this lemma, we obviously have

$$\int_S R(x) e^{u_k} dA \to \int_S R(x) e^{u_o} dA,$$

and it follows that $J(\cdot)$ is weakly lower semi-continuous. On the other hand, we have

$$J(u_o) \geq \inf_{u \in H_e(S)} J(u).$$

Therefore, u_o is a minimum of J in $H_e(S)$.

Consequently, for any $v \in H_e(S)$, we have

$$0 =< J'(u_o), v >= \int_S < \bigtriangledown u_o, \bigtriangledown v > dA - \frac{8\pi}{\int_S R(x) e^{u_o} dA} \int_S R(x) e^{u_o} v dA.$$

For any $w \in H(S)$, let

$$v(x) = \frac{1}{2}(w(x) + w(-x)) := \frac{1}{2}(w(x) + w_-(x)).$$

Then obviously, $v \in H_e(S)$, and it follows that

$$0 =< J'(u_o), v >= \frac{1}{2}(< J'(u_o), w > + < J'(u_o), w_- >) =< J'(u_o), w > .$$

Here we have used the fact that $u_o(-x) = u_o(x)$. This implies that u_o is a critical point of J in $H^1(S)$ and hence a constant multiple of u_o is a weak solution of equation (5.3). Now what left is to prove Lemma 5.1.4.

The Proof of Lemma 5.1.4.

From Lemma 5.1.2, for any real number $p > 0$, we have

$$\int_S e^{pu} dA \leq C \exp\{\frac{p^2}{16\pi} \int_S |\bigtriangledown u|^2 dA + \frac{p}{4\pi} \int_S u dA\}. \tag{5.23}$$

And by *Höder* inequality, for any $0 < p < 2$,

$$\int_S |\bigtriangledown(e^u)|^p dA = \int_S e^{pu} |\bigtriangledown u|^p dA \leq \left(\int_S e^{\frac{2p}{2-p}u} dA\right)^{\frac{2-p}{2}} \left(\int_S |\bigtriangledown u|^2 dA\right)^{\frac{p}{2}}. \tag{5.24}$$

Inequalities (5.23) and (5.24) imply that $\{e^{u_k}\}$ is a bounded sequence in $H^{1,p}$ for any $1 < p < 2$. By the compact Sobolev embedding of $H^{1,p}$ into L^1, there exists a subsequence of $\{e^{u_k}\}$ which converges strongly to e^{u_o} in $L^1(S)$. This completes the proof of the Lemma.

Chen and Ding [CD] generalized Moser's result to a broader class of symmetric functions, as we will describe in the following.

Let $O(3)$ be the group of orthogonal transformations of R^3. Let G be a closed finite subgroup of $O(3)$. Let

$$F_G = \{x \in S^2 \mid gx = x, \ \forall g \in G\}$$

be the set of fixed points under the action of G.

The action of G on S^2 induces an action of G on the Hilbert space $H^1(S)$:

$$g[u](x) \mapsto u(gx)$$

under which the fixed point subspace of $H^1(S)$ is given by

$$X = \{u \in H^1(S) \mid u(gx) = u(x), \ \forall g \in G\}.$$

Assume that $R(x)$ is G-symmetric, or G-invariant, i.e.

$$R(gx) = R(x), \quad \forall g \in G. \tag{5.25}$$

Let $X_* = X \bigcap H(S)$. Recall that

$$H(S) = \{u \in H^1(S) \mid \int_S u\, dA = 0, \int_S R(x) e^u dA > 0\}.$$

Under the assumption (5.25), the functional

$$J(u) = \frac{1}{2} \int_S |\bigtriangledown u|^2 \, dx - 8\pi \ln \int_S R(x) e^u \, dx$$

is G-invariant in X_*, i.e.

$$J(g[u]) = J(u), \ \ \forall g \in G, \ \ \forall u \in X_*.$$

We seek a minimum of J in X_*. The following lemma guarantees that such a minimum is the critical point we desired.

Lemma 5.1.5 *Assume that u_o is a critical point of J in X_*, then it is also a critical point of J in $H(S)$.*

Proof. First, for any $g \in G$ and any $u, v \in H(S)$, we have

$$< J'(u), g[v] >=< J'(g^{-1}[u]), v > . \tag{5.26}$$

Where g^{-1} is the inverse of g. This can be easily see from

$$< J'(u), v >= \int_S \bigtriangledown u \bigtriangledown v dA - \frac{8\pi}{\int_S Re^u dA} \int_S Re^u v dA.$$

Assume

$$G = \{g_1, \cdots g_m\}.$$

For any $w \in H(S)$, let

$$v = \frac{1}{m}(g_1[w] + \cdots + g_m[w]).$$

Then for any $g \in G$

$$g[v] = \frac{1}{m}(gg_1[w] + \cdots + gg_m[w]) = v.$$

Hence $v \in X_*$. It follows that

$$\begin{aligned} 0 =< J'(u_o), v >&= \frac{1}{m}\sum_i^m < J'(u_o), g_i[w] > \\ &= \frac{1}{m}\sum_i^m < J'(g_i^{-1}u_o), w >= \frac{1}{m}\sum_i^m < J'(u_o), w > \\ &=< J'(u_o), w > . \end{aligned}$$

Therefore, u_o is a critical point of J in $H(S)$. □

Theorem 5.1.2 *(Chen and Ding [CD]). suppose that $R \in C^\alpha(S^2)$ satisfies (5.25) with $\max_S R > 0$. If $F_G \neq \emptyset$, let $m = \max_{F_G} R$. Then equation (5.2) has a solution provided one of the following conditions holds:*

(i) $F_G = \emptyset$,
(ii) $m \leq 0$, *or*
(iii) $m > 0$ *and* $c_* = \inf_{X_*} J < -8\pi \ln 4\pi m$.

Remark 5.1.1 *In the case when the function $R(x)$ is even, the group $G = \{id, -id\}$, where id is the identity transform, i.e.*

$$id(x) = x \quad and \; - id(x) = -x \;\; \forall \, x \in S^2.$$

Obviously, in this case F_G is empty. Hence the above Theorem contains Moser's existence result as a special case.

To prove this theorem, we need another two lemmas.

Lemma 5.1.6 *(Onofri [On]) For any $u \in H^1(S)$ with $\int_S udA = 0$, holds*

$$\int_S e^u dA \leq 4\pi \exp\left\{\frac{1}{16\pi}\int_S |\bigtriangledown u|^2 dA\right\}. \tag{5.27}$$

Lemma 5.1.7 *Suppose that $\{u_i\}$ is a sequence in $H(S)$ with $J(u_i) \leq \beta$. If $\|u_i\| \rightarrow \infty$, then there exists a subsequence (still denoted by $\{u_i\}$) and a point $\xi \in S^2$ with $R(\xi) > 0$, such that*

$$\int_S R(x) e^{u_i} dA = (R(\xi) + o(1)) \int_S e^{u_i} dA, \tag{5.28}$$

and

$$\lim_{i \rightarrow \infty} J(u_i) \geq -8\pi \ln 4\pi R(\xi). \tag{5.29}$$

Proof. We first show that, there exists a subsequence of $\{u_i\}$ and a point $\xi \in S^2$, such that for any $\epsilon > 0$, we have

$$\int_{S \setminus B_\epsilon(\xi)} e^{u_i} dA \rightarrow 0. \tag{5.30}$$

Otherwise, there exists two subsets of S^2, Ω_1 and Ω_2, with dist$(\Omega_1, \Omega_2) \geq \epsilon_o > 0$ such that $\{u_i\}$ satisfies all the conditions in Lemma 5.1.3. Hence the inequality

$$\int_S e^{u_i} dA \leq C \exp(\frac{1}{16\pi} - \delta)\|u_i\|^2$$

holds for some $\delta > 0$. Then by the previous argument, we conclude that $\|u_i\|$ is bounded. This contradicts with our assumption, and hence verifies (5.30). Now (5.30) and the continuity assumption on $R(x)$ implies (5.28).

Roughly speaking, the above argument shows that if a minimizing sequence of J is unbounded, then passing to a subsequence, the mass of the surface S with density $e^{u_i(x)}$ at point x will be concentrating at one point ξ on S.

To verify (5.29), we use (5.28) and Onofri's Lemma:

$$\begin{aligned}
J(u_i) &= \frac{1}{2}\int_S |\bigtriangledown u|^2 dA - 8\pi \ln (R(\xi) + o(1)) \int_S e^{u_i} dA \\
&= \frac{1}{2}\int_S |\bigtriangledown u|^2 dA - 8\pi \ln (R(\xi) + o(1)) - 8\pi \ln \int_S e^{u_i} dA \\
&\geq \frac{1}{2}\int_S |\bigtriangledown u|^2 dA - 8\pi \ln (R(\xi) + o(1)) - 8\pi(\ln 4\pi + \frac{1}{16\pi}\int_S |\bigtriangledown u|^2 dA) \\
&= -8\pi \ln (4\pi R(\xi) + o(1)).
\end{aligned}$$

Let $i {\rightarrow} \infty$, we arrive at (5.29). Since the right hand side of (5.29) is bounded above by β, we must have $R(\xi) > 0$. This completes the proof of the Lemma.

Proof of the Theorem. Define

$$c_* = \inf_{X_*} J.$$

As we argue in the previous section (see (5.12)), we have $c_* > -\infty$. Let $\{u_i\} \subset X_*$ be a minimizing sequence, i.e. $J(u_i){\rightarrow}c_*$, as $i{\rightarrow}\infty$.

Similar to the reasoning at the beginning of this section, one can see that we only need to show that $\{u_i\}$ is bounded in $H^1(S)$. Assume on the contrary that $\|u_i\|{\rightarrow}\infty$. Then from the proof of Lemma 5.1.7, there is a point $\xi \in S^2$, such that for any $\epsilon > 0$, we have

$$\int_{S \setminus B_\epsilon(\xi)} e^{u_i} dA {\rightarrow} 0. \tag{5.31}$$

Since u_i is invariant under the action of G, we must also have

$$\int_{S \setminus B_\epsilon(g\xi)} e^{u_i} dA {\rightarrow} 0. \tag{5.32}$$

Because ϵ is arbitrary, we deduce that $g\xi = \xi$, that is $\xi \in F_G$.

Now under condition (i), F_G is empty. Therefore, $\{u_i\}$ must be bounded.

Moreover, by Lemma 5.1.7,

$$R(\xi) > 0 \quad \text{and} \quad c_* \geq -8\pi \ln 4\pi R(\xi).$$

Noticing that $m \geq R(\xi)$ by definition, this again contradicts with conditions (ii) and (iii).

Therefore, under either one of the conditions (i), (ii), or (iii), the minimizing sequence $\{u_i\}$ must be bounded, and hence possesses a subsequence converging weakly to some $u_o \in X_*$. This a constant multiple of u_o is the desired weak solution of equation (5.3).

Here conditions (i) and (ii) in Theorem 5.1.2 can be easily verified through the group G or the given function R. However, one may wonder, under what conditions on R, do we have condition (iii). The following theorem provides a sufficient condition.

Theorem 5.1.3 *(Chen and Ding [CD]). Suppose that $R \in C^2(S^2)$, is G-invariant, and $m = \max_{F_G} R > 0$. Let*

$$D = \{x \in S^2 \mid R(x) = m\}.$$

If $\triangle R(x_o) > 0$ for some $x_o \in D$, then

$$c_* < -8\pi \ln 4\pi m, \tag{5.33}$$

and therefore (5.3) possesses a weak solution.

Proof. Choose a spherical polar coordinate system on $S := S^2$:

$$(\theta, \phi), \ \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ \ 0 \le \phi \le 2\pi$$

with $x_o = (\frac{\pi}{2}, \phi)$ as the north pole.

Consider the following family of functions:

$$u_\lambda(\theta, \phi) = \ln \frac{1-\lambda^2}{(1-\lambda \sin\theta)^2}, \ \ v_\lambda = u_\lambda - \frac{1}{4\pi}\int_S u_\lambda dA, \ \ 0 < \lambda < 1.$$

One can verify that this family of functions satisfy the equation

$$-\triangle u_\lambda + 2 = 2e^{u_\lambda}. \tag{5.34}$$

As $\lambda \to 1$, the family $\{u_\lambda\}$ 'blows up' (tend to ∞) at the one point x_o, while approaching $-\infty$ everywhere else on S^2. Therefore, the mass of the sphere with density $e^{u_\lambda(x)}$ is concentrating at point x_o as $\lambda \to 1$.

It is easy to see that $v_\lambda \in H(S)$ for λ closed to 1. Moreover, we claim that

$$v_\lambda \in X, \ \text{ i.e. } \ v_\lambda(gx) = v_\lambda(x), \ \forall g \in G.$$

In fact, since g is an isometry of S^2, and $gx_o = x_o$, we have

$$d(gx, x_o) = d(gx, gx_o) = d(x, x_o),$$

where $d(\cdot, \cdot)$ is the geodesic distance on S^2. But v_λ depends only on $\theta = d(x, x_o)$, so

$$v_\lambda(gx) = v_\lambda(x), \ \ \forall g \in G.$$

It follows that $v_\lambda \in X_*$ for λ sufficiently closed to 1.

By a direct computation, one can verify that

$$\frac{1}{2}\int_S |\bigtriangledown u_\lambda|^2 dA + 8\pi \bar{u}_\lambda = 0.$$

Consequently,

$$J(v_\lambda) = -8\pi \ln \int_S R(x) e^{u_\lambda} dA.$$

Notice that $c_* \le J(v_\lambda)$ by the definition of c_*. Thus, in order to show (5.33), we only need to verify that, for λ sufficiently close to 1

$$\int_S R(x) e^{u_\lambda} dA > 4\pi m. \tag{5.35}$$

Let $\delta > 0$ be small, we have

$$\begin{aligned}\int_S Re^{u_\lambda} dA = (1-\lambda^2)\Bigg\{&\int_0^{2\pi}\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} [R(\theta,\phi) - m]\frac{\cos\theta}{(1-\lambda\sin\theta)^2} d\theta d\phi \\ &+ \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}-\delta} [R(\theta,\phi) - m]\frac{\cos\theta}{(1-\lambda\sin\theta)^2} d\theta d\phi\Bigg\} + 4\pi m \\ = (1-\lambda^2)\{I + II\} + 4\pi m.&\end{aligned}$$

Here we have used the fact

$$\int_S e^{u_\lambda} dA = 4\pi$$

which can be derived directly from equation (5.34).

For each fixed δ, it is easy to check that the integral II remains bounded as λ tends to 1. Thus (5.35) will hold if we can show that the integral I $\to +\infty$ as $\lambda \to 1$.

Notice that x_o is a maximum point of the restriction of R on F_G. Hence by the principle of symmetric critically, x_o is a critical point of R, that is $\bigtriangledown R(x_o) = 0$. Using this fact and the second order Taylor expansion of $R(x)$ near x_o, we can derive

$$\begin{aligned} I &= \pi \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \frac{\left[\triangle R(x_o)\cos^2\theta + o\left(\left|\theta - \frac{\pi}{2}\right|^2\right)\right]}{(1-\lambda\sin\theta)^2} \cos\theta \, d\theta \\ &\geq c_o \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \frac{\cos^3\theta}{(1-\lambda\sin\theta)^2} d\theta. \end{aligned}$$

Here c_o is a positive constant. We have chosen δ to be sufficiently small and have used the fact that $\triangle R(x_o)$ is positive. From the above inequality, one can easily verify that as $\lambda \to 1, I \to +\infty$, since the integral

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \frac{\cos^3\theta}{(1-\sin\theta)^2} d\theta$$

diverges to $+\infty$.

This completes the proof of the Theorem. □

5.2 Prescribing Gaussian Curvature on Negatively Curved Manifolds

Let M be a compact two dimensional Riemannian manifold. Let $g_o(x)$ be a metrics on M with the corresponding Laplace-Beltrami operator $\triangle$ and Gaussian curvature $K_o(x)$. Given a function $K(x)$ on M, can it be realized as the Gaussian curvature associated to the point-wise conformal metrics $g(x) = e^{2u(x)} g_o(x)$? As we mentioned before, to answer this question, it is equivalent to solve the following semi-linear elliptic equation

$$-\triangle u + K_o(x) = K(x)e^{2u} \quad x \in M. \tag{5.36}$$

To simplify this equation, we introduce the following simple existence result.

Lemma 5.2.1 *Let $f(x) \in L^2(M)$. Then the equation*

$$-\triangle u = f(x) \;\; x \in M \tag{5.37}$$

possesses a solution if and only if

$$\int_M f(x) d A = 0. \tag{5.38}$$

Proof. The 'only if' part can be seen by integrating both side of the equation (5.37) on M.

We now prove the 'if' part. Assume that (5.38) holds, we will obtain the existence of a solution to (5.37). To this end, we consider the functional

$$J(u) = \frac{1}{2} \int_M |\bigtriangledown u|^2 d A - \int_M f(x) u d A.$$

under the constraint

$$G = \{u \in H^1(M) \mid \int_M u(x) d A = 0\}.$$

As we have seen before, we can use, in G, the equivalent norm

$$\|u\| = \left(\int_M |\bigtriangledown u|^2 d A \right)^{\frac{1}{2}}.$$

By *Höder* and Poincaré inequalities, we derive

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|u\|^2 - \|f\|_{L^2} \|u\|_{L^2} \\ &\geq \frac{1}{2} \|u\|^2 - c_o \|f\|_{L^2} \|u\| \\ &= \frac{1}{2} (\|u\| - c_o \|f\|_{L^2})^2 - \frac{c_o^2 \|f\|_{L^2}^2}{2}. \end{aligned} \tag{5.39}$$

The above inequality implies immediately that the functional $J(u)$ is bounded from below in the set G.

Let $\{u_k\}$ be a minimizing sequence of J in G, i.e.

$$J(u_k) \to \inf_G J, \quad \text{as } k \to \infty.$$

Then again by (5.39), $\{u_k\}$ is bounded in H^1, and hence possesses a subsequence (still denoted by $\{u_k\}$) that converges weakly to some u_o in $H^1(M)$. From compact Sobolev imbedding

$$H^1 \hookrightarrow\hookrightarrow L^2,$$

we see that

$$u_k {\rightarrow} u_o \quad \text{strongly in } L^2, \quad \text{and hence so in} \quad L^1.$$

Therefore

$$\int_M u_o(x) d\,A = \lim \int_M u_k(x) d\,A = 0, \tag{5.40}$$

and

$$\int_M f(x) u_o(x) d\,A = \lim \int_M f(x) u_k(x) d\,A. \tag{5.41}$$

(5.40) implies that $u_o \in G$, and hence

$$J(u_o) \geq \inf_G J(u). \tag{5.42}$$

While (5.41) and the weak lower semi-continuity of the Dirichlet integral:

$$\liminf \int_M |\bigtriangledown u_k|^2 d\,A \geq \int_M |\bigtriangledown u_o|^2 d\,A$$

leads to

$$J(u_o) \leq \inf_G J(u).$$

From this and (5.42) we conclude that u_o is the minimum of J in the set G, and therefore there exists constant λ, such that u_o is a weak solution of

$$-\triangle u_o - f(x) = \lambda.$$

Integrating both sides and by (5.38), we find that $\lambda = 0$. Therefore u_o is a solution of equation (5.37). This completes the proof of the Lemma. □

We now simplify equation (5.36). First let $\tilde{u} = 2u$, then

$$-\triangle \tilde{u} + 2K_o(x) = 2K(x) e^{\tilde{u}}. \tag{5.43}$$

Let $v(x)$ be a solution of

$$-\triangle v = 2K_o(x) - 2\bar{K_o} \tag{5.44}$$

where

$$\bar{K_o} = \frac{1}{|M|} \int_M K_o(x) d\,A$$

is the average of $K_o(x)$ on M. Because the integral of the right hand side of (5.44) on M is 0, the solution of (5.44) exists due to Lemma 5.2.1.

Let $w = \tilde{u} + v$. Then it is easy to verify that

$$-\triangle w + 2\bar{K_o} = 2K(x) e^{-v} e^{w}. \tag{5.45}$$

Finally, letting

$$\alpha = 2\bar{K}_o \quad \text{and } R(x) = 2K(x)e^{-v(x)}$$

and rewriting w as u, we reduce equation (5.36) to

$$-\triangle u + \alpha = R(x)e^{u(x)}\,, \quad x \in M \tag{5.46}$$

By the well-known Gauss-Bonnet Theorem, if g is a Riemannian metrics on M with corresponding Gaussian curvature $K_g(x)$, then

$$\int_M K_g(x)d\,A_g = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler Characteristic of M, and $d\,A_g$ the area element associated to metric g.

We say M is negatively curved, if $\chi(M) < 0$, and hence $\alpha < 0$.

5.2.1 Kazdan and Warner's Results. Method of Lower and Upper Solutions

Let M be a compact two dimensional manifold. Let $\alpha < 0$. In [KW], Kazdan and Warner considered the equation

$$-\triangle u + \alpha = R(x)e^{u(x)}\,, \quad x \in M \tag{5.47}$$

and proved

Theorem 5.2.1 *(Kazdan-Warner) Let $\bar{R}$ be the average of $R(x)$ on M. Then*

i) A necessary condition for (5.46) to have a solution is $\bar{R} < 0$.

ii) If $\bar{R} < 0$, then there is a constant α_o, with $-\infty \leq \alpha_o < 0$, such that one can solve (5.46) if $\alpha > \alpha_o$, but cannot solve (5.46) if $\alpha < \alpha_o$.

iii) $\alpha_o = -\infty$ if and only if $R(x) \leq 0$.

The existence of a solution was established by using the method of upper and lower solutions. We say that u_+ is an upper solution of equation (5.47) if

$$-\triangle u_+ + \alpha \geq R(x)e^{u_+}\,, \;\; x \in M.$$

Similarly, we say that u_- is a lower solution of (5.47) if

$$-\triangle u_- + \alpha \leq R(x)e^{u_-}\,, \;\; x \in M.$$

The following approach is adapted from [KW] with minor modifications.

Lemma 5.2.2 *If a solution of (5.47) exists, then $\bar{R} < 0$. Even more strongly, the unique solution ϕ of*

$$\triangle\phi + \alpha\phi = R(x)\,, \;\; x \in M \tag{5.48}$$

must be positive.

Proof. We first prove the second part. Using the substitution $v = e^{-u}$, one sees that (5.47) has a solution u if and only if there is a positive solution v of

$$\triangle v + \alpha v - R(x) - \frac{|\nabla v|^2}{v} = 0. \tag{5.49}$$

Assume that v is such a positive solution. Let ϕ be the unique solution of (5.48) and let $w = \phi - v$. Then

$$\triangle w + \alpha w = -\frac{|\nabla v|^2}{v} \leq 0. \tag{5.50}$$

It follows that $w \geq 0$. Otherwise, there is a point $x_o \in M$, such that $w(x_o) < 0$ and x_o is a minimum of w, hence $\triangle w(x_o) \geq 0$. Consequently, we must have

$$\triangle w(x_o) + \alpha w(x_o) > 0.$$

A contradiction with (5.50). Therefore $w \geq 0$ and $\phi \geq v > 0$. This shows that a necessary condition for (5.47) to have a solution is that the unique solution ϕ of (5.48) must be positive. Integrating both sides of (5.48), we see immediately that $\bar{R} < 0$. This completes the proof of the Lemma. □

Lemma 5.2.3 *(The Existence of Upper Solutions).*

i) If $R(x) \leq 0$ but not identically 0, then there exist an upper solution u_+ of (5.47).

ii) If $\bar{R} < 0$, then there exists a small $\delta > 0$, such that for $-\delta \leq \alpha < 0$, there exists an upper solution u_+ of (5.47).

Proof. Let v be a solution of

$$-\triangle v = R(x) - \bar{R}.$$

We try to choose constant a and b, such that $u_+ = av + b$ is an upper solution. We have

$$-\triangle u_+ + \alpha - R(x)e^{u_+} = aR(x) - a\bar{R} + \alpha - R(x)e^{av+b}. \tag{5.51}$$

We want to choose constant a and b, so that the right hand side of (5.51) is non-negative.

In Case i), when $R(x) \leq 0$, we regroup the right hand side of (5.51) as

$$f(a, b, x) \equiv (-a\bar{R} + \alpha) - R(x)(e^{av+b} - a). \tag{5.52}$$

For any given $\alpha < 0$, we first choose a sufficiently large, so that $-a\bar{R} + \alpha > 0$. Then we choose b large, so that

$$e^{av(x)+b} - a \geq 0\,, \quad \forall\, x \in M.$$

This is possible because $v(x)$ is continuous and M is compact. It follows from (5.51) and (5.52) that

$$-\triangle u_+ + \alpha - R(x)e^{u_+} \geq 0.$$

In Case ii), we choose $\alpha \geq \frac{a\bar{R}}{2}$ and $e^b = a$, where a is a positive number to be determine later. Then (5.52) becomes

$$\begin{aligned} f(a,b,x) &\geq -\frac{a\bar{R}}{2} - aR(x)(e^{av} - 1) \\ &\geq a\left(-\frac{\bar{R}}{2} - \|R\|_{L^\infty}|e^{av} - 1|\right) \\ &= a\|R\|_{L^\infty}\left(\frac{-\bar{R}}{2\|R\|_{L^\infty}} - |e^{av} - 1|\right). \end{aligned}$$

Again by the continuity of v and the compactness of M, we see that

$$e^{av(x)} \to 1\,, \quad \text{uniformly on } M \text{ as } a \to 0.$$

Therefore, we can make

$$|e^{av(x)} - 1| \leq \frac{-\bar{R}}{2\|R\|_{L^\infty}}\,, \quad \text{for all } x \in M,$$

by choosing a sufficiently small. Thus u_+ so constructed is an upper solution of (5.47) for $\alpha \geq \frac{a\bar{R}}{2}$. This completes the proof of the Lemma. □

Lemma 5.2.4 *(Existence of Lower Solutions) Given any $R \in L^p(M)$ and a function $u_+ \in W^{2,p}(M)$, there is a lower solution $u_- \in W^{2,p}(M)$ of (5.47) such that $u_- < u_+$.*

Proof. If $R(x)$ is bounded from below, then one can clearly use any sufficiently large negative constant as u_-. For general $R(x) \in L^p(M)$, let $K(x) = \max\{1, -R(x)\}$. Choose a positive constant a so that $a\bar{K} = -\alpha$. Then $\overline{aK(x) + \alpha} = 0$ and $(aK(x) + \alpha) \in L^p(M)$. Thus there is a solution w of

$$\triangle w = aK(x) + \alpha.$$

By the L_p regularity theory, $w \in W^{2,p}(M)$ and hence is continuous. Let $u_- = w - \lambda$. Then

$$\begin{aligned} -\triangle u_- + \alpha - R(x)e^{u_-} &= -\triangle w + \alpha - R(x)e^{w-\lambda} \\ &= -aK(x) - R(x)e^{w-\lambda} \\ &\leq -K(x)(a - e^{w-\lambda}). \end{aligned} \tag{5.53}$$

Choosing λ sufficiently large so that

$$a - e^{w-\lambda} \geq 0,$$

and noticing that $K(x) \geq 0$, by (5.53), we arrive at

$$-\triangle u_- + \alpha - R(x)e^{u_-} \leq 0.$$

Therefore u_- is a lower solution. Moreover, one can make λ large enough such that $u_- = w - \lambda < u_+$. This completes the proof of the Lemma. □

Lemma 5.2.5 *Let $p > 2$. If there exist upper and lower solutions, $u_+, u_- \in W^{2,p}(M)$ of (5.47), and if $u_- \leq u_+$, then there is a solution $u \in W^{2,p}(M)$. Moreover, $u_- \leq u \leq u_+$, and u is C^∞ in any open set where $R(x)$ is C^∞.*

Proof. We will rewrite the equation (5.47) as

$$Lu = f(x, u),$$

then start from the upper solution u_+ and apply iterations:

$$Lu_1 = f(x, u_+) \ , Lu_{i+1} = f(x, u_i) \, , \ \ i = 1, 2, \cdots .$$

In order that for each i, $u_- \leq u_i \leq u_+$, we need the operator L obeys some maximum principle and $f(x, \cdot)$ possess some monotonicity. The Laplace operator in the original equation does not obey maximum principles, however, if we let

$$Lu = -\triangle u + k(x)u,$$

where $k(x)$ is any positive function on M, then the new operator L obeys the maximum principle:

$$\text{If } Lu \geq 0, \text{ then } u(x) \geq 0 \, , \quad \text{for all } x \in M.$$

To see this, we suppose in the contrary, there is some point on M where $u < 0$. Let x_o be a negative minimum of u, then $-\triangle u(x_o) \leq 0$, and it follows that

$$-\triangle u(x_o) + k(x_o)u(x_o) \leq k(x_o)u(x_o) < 0.$$

A contradiction. Hence the maximum principle holds for L. We now write equation (5.47) as

$$Lu \equiv -\triangle u + k(x)u = R(x)e^u - \alpha + k(x)u := f(x, u).$$

Based on the maximum principle on L, to keep $u_- \leq u_i \leq u_+$, it requires that $f(x, \cdot)$ be monotone increasing, that is

$$\frac{\partial f}{\partial u} = R(x)e^u + k(x) \geq 0.$$

To this end, we choose

$$k(x) = k_1(x)e^{u_+(x)} \, , \quad \text{where} \quad k_1(x) = \max\{1, -R(x)\}.$$

Base on this choice of L and f, we show by math induction that the iteration sequence so constructed is monotone decreasing:

$$u_{i+1} \leq u_i \leq \cdots \leq u_1 \leq u_+ \,, \quad i = 1, 2, \cdots . \tag{5.54}$$

In fact, from

$$Lu_+ \geq f(x, u_+) \quad \text{ and } \quad Lu_1 = f(x, u_+)$$

we derive immediately

$$L(u_+ - u_1) \geq 0.$$

Then the maximum principle for L implies $u_+ \geq u_1$. This completes the first step of our induction. Assume that for any integer m, $u_{m+1} \leq u_m \leq u_+$, we will show that $u_{m+2} \leq u_{m+1}$. Since $f(x, \cdot)$ is monotone increasing in u for $u \leq u_+$, we have

$$f(x, u_{m+1}) \leq f(x, u_m)$$

and hence $L(u_{m+1} - u_{m+2}) \geq 0$. Therefore $u_{m+1} \geq u_{m+2}$. This verifies (5.54) through math induction. Similarly, one can show the other side of the inequality and arrive at

$$u_- \leq u_{i+1} \leq u_i \leq \cdots \leq u_+ \,, \quad i = 1, 2, \cdots . \tag{5.55}$$

Since u_+, u_-, and u_i are continuous, inequality (5.55) shows that $\{|u_i|\}$ is uniformly bounded. Consequently, in the L^p norm,

$$\|Lu_{i+1}\|_{L^p} = \|R(x)e^{u_i} - \alpha + k(x)u_i\|_{L^p} \leq C.$$

Hence $\{u_i\}$ is bounded in $W^{2,p}(M)$. By the compact imbedding of $W^{2,p}$ into C^1, there is a subsequence that converges uniformly to some function u in $C^1(M)$. In view of the monotonicity (5.55), we conclude that the entire sequence $\{u_i\}$ itself converges uniformly to u. Moreover

$$\begin{aligned}\|u_{i+1} - u_{j+1}\|_{W^{2,p}} &\leq C\|L(u_{i+1} - u_{j+1})\|_{L^p} \\ &\leq C(\|R\|_{L^p}\|e^{u_i} - e^{u_j}\|_{L^\infty} + \|k\|_{L^p}\|u_i - u_j\|_{L^\infty}).\end{aligned}$$

Therefore, $\{u_i\}$ converges strongly in $W^{2,p}$, so $u \in W^{2,p}$. Since $L : W^{2,p} \rightarrow L^p$ is continuous, it follows that u is a solution of (5.47) and satisfies $u_- \leq u \leq u_+$.

By the Schauder theory, one proves inductively that $u \in C^\infty$ in any open set where $R(x)$ is C^∞. More precisely, if $R \in C^{m+\gamma}$, then $u \in C^{m+2+\gamma}$. This completes the proof of the Lemma. □

Based on these Lemmas, we are now able to complete the proof of the Theorem 5.2.1.

Proof of Theorem 5.2.1.

Part i) can obviously derived from Lemma 5.2.2.

From Lemma 5.2.4 and 5.2.5, if there is a upper solution, then there is a solution. Hence by Lemma 5.2.3, there exist a $\delta > 0$, such that (5.47) is solvable for $-\delta \leq \alpha < 0$. Let

$$\alpha_o = \inf\{\alpha \mid (5.47) \text{ is sovable }\}.$$

Then for any $0 > \alpha > \alpha_o$, one can solve (5.47). In fact, if for α_1, (5.47) possesses a solution u_1, then for any $\alpha > \alpha_1$, we have

$$-\triangle u_1 + \alpha > R(x)e^{u_1},$$

that is, u_1 is an upper solution for equation (5.47) at α. Hence equation

$$-\triangle u + \alpha = R(x)e^{u}$$

also has a solution.

Lemma 5.2.3 infers that $\alpha_o = -\infty$ if $R(x) \leq 0$ but $R(x)$ is not identically 0. Now to complete the proof of the Theorem, it suffice to prove that if $R(x)$ is positive somewhere, then $\alpha_o > -\infty$. By virtue of Lemma 5.2.2, we only need to show

Claim: There exist $\alpha < 0$, such that the unique solution of

$$\triangle \phi + \alpha \phi = R(x) \tag{5.56}$$

is negative somewhere.

Actually, we can prove a stronger result:

$$-\alpha\phi(x) \rightarrow R(x) \quad \text{almost everywhere as } \alpha \rightarrow -\infty. \tag{5.57}$$

We argue in the following 2 steps.

i) For a given $\alpha < 0$, let $u = u(x, \alpha)$ be the solution of (5.56). Multiplying both sides of equation (5.56) by u and integrate over M to obtain

$$-\int_M |\bigtriangledown u|^2 dA + \alpha \int_M u^2 dA = \int_M R(x)u(x)dA.$$

Consequently,

$$\|u\|_{L^2} \leq \frac{1}{\alpha}\int_M RudA.$$

Then by *Höder* inequality

$$\|u\|_{L^2} \leq \frac{1}{\alpha}\|R\|_{L^2}.$$

This implies that

$$\text{as } \alpha \rightarrow -\infty, \; u(x,\alpha) \rightarrow 0, \quad \text{almost everywhere .} \tag{5.58}$$

Since for any $\alpha < 0$, the operator $\triangle + \alpha$ is invertible, we can express

$$u = (\triangle + \alpha)^{-1} R(x).$$

From the above reasoning, one can see that, as $\alpha \to -\infty$,

$$\text{if } R \in L_2, \;\; \text{then } (\triangle + \alpha)^{-1} R(x) \to 0 \;\text{ almost everywhere .} \tag{5.59}$$

ii) To see (5.57), we write

$$-\alpha\phi + R(x) = (\triangle + \alpha)^{-1} \triangle R. \tag{5.60}$$

One can see the validity of (5.60) by simply apply the operator $\triangle + \alpha$ to both sides.

Now by the assumption, we have $\triangle R \in L_2$, and hence (5.59) implies that

$$-\alpha\phi + R(x) \to 0, \; a.e..$$

Or

$$\phi(x) \to \frac{1}{\alpha} R(x). \; a.e.$$

Since $R(x)$ is continuous and positive somewhere, for sufficiently negative α, $\phi(x)$ is negative somewhere.

This completes the proof of the Theorem.

5.2.2 Chen and Li's Results

From the previous section, one can see from the Kazdan and Warner's result that in the case when $R(x)$ is non-positive, equation (5.46) has been solved completely. i.e. a necessary and sufficient condition for equation (5.46) to have a solution for any $0 > \alpha > -\infty$ is $R(x) \leq 0$. However, in the case when $R(x)$ changes signs, we have $\alpha_o > -\infty$, and hence there are still some questions remained. In particular, one may naturally ask:

Does equation (5.46) has a solution for the critical value $\alpha = \alpha_o$?

In [CL1], Chen and Li answered this question affirmatively.

Theorem 5.2.2 *(Chen-Li)*

Assume that $R(x)$ is a continuous function on M. Let α_o be defined in the previous section. Then equation (5.46) has at least one solution when $\alpha = \alpha_o$.

Proof. The Outline. Let $\{\alpha_k\}$ be a sequence of numbers such that

$$\alpha_k > \alpha_o, \;\; \text{and } \alpha_k \to \alpha_o, \;\; \text{as } \; k \to \infty.$$

We will prove the Theorem in 3 steps.

In step 1, we minimize the functional, for each fixed α_k,

$$J_k(u) = \frac{1}{2} \int_M |\nabla u|^2 d\,A + \alpha_k \int_M u d\,A - \int_M R(x) e^u d\,A$$

in a class of functions that are between a lower and a Upper solution. Let the minimizer be u_k. Then let $\alpha_k \to \alpha_o$.

In step 2, we show that the sequence of minimizers $\{u_k\}$ is bounded in the region where $R(x)$ is positively bounded away from 0.

In step 3, we prove that $\{u_k\}$ is bounded in $H^1(M)$ and hence converges to a desired solution.

Step 1. For each $\alpha_k > \alpha_o$, by Kazdan and Warner, there exists a solution ϕ_k of

$$-\triangle \phi_k + \alpha_k = R(x) e^{\phi_k}. \tag{5.61}$$

We first show that there exists a constant $c_o > 0$, such that

$$\phi_k(x) \geq -c_o \,, \forall\, x \in M \;\text{ and } \forall\, k = 1, 2, \cdots. \tag{5.62}$$

Otherwise, for each k, let x^k be a minimum of $\phi_k(x)$ on M. Then obviously,

$$\phi_k(x^k) \to -\infty \,, \text{ as } k \to \infty.$$

On the other hand, since x^k is a minimum, we have

$$-\triangle \phi_k(x^k) \leq 0 \,, \forall\, k = 1, 2, \cdots.$$

These contradict equation (5.61) because

$$\alpha_k \to \alpha_o < 0 \,, \text{ as } k \to \infty.$$

This verifies (5.62).

Choose a sufficiently negative constant $A < -c_o$ to be the lower solution of equation (5.46) for all α_k, that is,

$$-\triangle A + \alpha_k \leq R(x) e^A.$$

This is possible because $R(x)$ is bounded below on M, and hence we can make $R(x)e^A$ greater than any fixed negative number.

For each fixed α_k, choose $\alpha_o < \tilde{\alpha}_k < \alpha_k$. By the result of Kazdan and Warner, there exists a solution of equation (5.46) for $\alpha = \tilde{\alpha}_k$. Call it ψ_k. Then

$$-\triangle \psi_k + \alpha_k > -\triangle \psi_k + \tilde{\alpha}_k = R(x) e^{\psi_k(x)},$$

i.e. ψ_k is a super solution of the equation

$$-\triangle u + \alpha_k = R(x) e^{u(x)}. \tag{5.63}$$

Illuminated by an idea of Ding and Liu [DL], we minimize the functional $J_k(u)$ in a class of functions

$$H = \{u \in C^1(M) \mid A \leq u \leq \psi_k\}.$$

This is possible, because the functional $J_k(u)$ is bounded from below. In fact, since M is compact and $R(x)$ is continuous, there exists a constant C_1, such that

$$|R(x)| \leq C_1, \quad \forall\, x \in M.$$

Consequently, for any $u \in H$,

$$\begin{aligned} J_k(u) &\geq \frac{1}{2}\int_M |\bigtriangledown u|^2 d\,A + \alpha_k \int_M u d\,A - C_1 \int_M e^{\psi_k} d\,A \\ &\geq \frac{1}{2}(\|u\| - C_2)^2 - C_3. \end{aligned} \tag{5.64}$$

Here we have used the *Hölder* and Poincaré inequality. From (5.64), one can easily see that J_k is bounded from below. Also by (5.65) and a similar argument as in the previous section, we can show that there is a subsequence of a minimizing sequence which converges weakly to a minimum u_k of $J_k(u)$ in the set H. In order to show that u_k is a solution of equation (5.63), we need it be in the interior of H, i.e.

$$A < u_k(x) < \psi_k(x), \quad \forall\, x \in M. \tag{5.65}$$

To verify this, we use a maximum principle.

First we show that

$$u_k(x) < \psi_k(x)\,, \quad \forall\, x \in M. \tag{5.66}$$

Suppose in the contrary, there exists $x_o \in M$, such that $u_k(x_o) = \psi_k(x_o)$, we will derive a contradiction. Since $u_k(x_o) > A$, there is a $\delta > 0$, such that

$$A < u_k(x)\,, \quad \forall x \in \overline{B_\delta(x_o)}.$$

It follows that, for any positive function $v(x)$ with its support in $B_\delta(x_o)$, there exists an $\epsilon > 0$, such that as $-\epsilon \geq t \leq 0$, we have $u_k + tv \in H$. Since u_k is a minimum of J_k in H, we have in particular that the function $g(t) = J_k(u_k + tv)$ achieves its minimum at $t = 0$ on the interval $[-\epsilon, 0]$. Hence $g'(0) \leq 0$. This implies that

$$\int_M [-\triangle u_k + \alpha_k - R(x)e^{u_k}]v(x)d\,A \leq 0. \tag{5.67}$$

Consequently

$$-\triangle u_k(x_o) + \alpha_k - R(x_o)e^{u_k(x_o)} \leq 0. \tag{5.68}$$

Let $w(x) = \psi_k(x) - u_k(x)$. Then $w(x) \geq 0$, and x_o is a minimum of w. It follows that $-\triangle w(x_o) \leq 0$. While on the other hand, we have, by (5.68),

$$-\triangle w(x_o) = -\triangle \psi_k(x_o) - (-\triangle u_k(x_o)) \geq -\tilde{\alpha}_k + \alpha_k > 0.$$

Again a contradiction. Therefore we must have

$$u_k(x) < \psi_k(x)\,, \quad \forall\, x \in M.$$

Similarly, one can show that

$$A < u_k(x)\,, \quad \forall\, x \in M.$$

This verifies (5.65). Now for any $v \in C^1(M)$, $u_k + tv$ is in H for all sufficiently small t, and hence $\frac{d}{dt} J_k(u + tv)|_{t=0} = 0$. Then a direct computation as we did in the past will lead to

$$-\triangle u_k + \alpha_k = R(x) e^{u_k}\,, \quad \forall\, x \in M.$$

Step 2. It is obvious that the sequence $\{u_k\}$ so obtained in the previous step is uniformly bounded from below by the negative constant A. To show that it is also bounded from above, we first consider the region where $R(x)$ is positively bounded away from 0. We will use a sup + inf inequality of Brezis, Li, and Shafrir [BLS]:

Lemma 5.2.6 *(Brezis, Li, and Shafrir). Assume that V is a Lipschitz function satisfying*

$$0 < a \leq V(x) \leq b < \infty$$

and let K be a compact subset of a domain Ω in R^2. Then any solution u of the equation

$$-\triangle u = V(x) e^u\,, x \in \Omega,$$

satisfies

$$\sup_K u + \inf_\Omega u \leq C(a, b, \|\bigtriangledown V\|_{L^\infty}, K, \Omega). \tag{5.69}$$

Let x^o be a point on M at which $R(x^o) > 0$. Choose ϵ so small such that

$$R(x) \geq a > 0\,, \quad \forall\, x \in \Omega \equiv B_{2\epsilon}(x^o),$$

for some constant a.

Let v_k be a solution of

$$\begin{cases} -\triangle v_k - \alpha_k = 0 & x \in \Omega \\ v_k(x) = 1 & x \in \partial\Omega. \end{cases}$$

Let $w_k = u_k + v_k$. Then w_k satisfies

$$-\triangle w_k = R(x) e^{-v_k} e^{w_k}.$$

By the maximum principle, the sequence $\{v_k\}$ is bounded from above and from below in the small ball Ω. Since $\{u_k\}$ is uniformly bounded from below, $\{w_k\}$ is also bounded from below. Locally, the metric is point-wise conformal to the Euclidean metric, hence one can apply Lemma 5.2.6 to $\{w_k\}$ to conclude that the sequence $\{w_k\}$ is also uniformly bounded from above in the smaller ball $B_\epsilon(x^o)$. Since x^o is an arbitrary point where R is positive, we conclude that $\{w_k\}$ is uniformly bounded in the region where $R(x)$ is positively bounded away from 0. Now the uniform bound for $\{u_k\}$ in the same region follows suit.

Step 3. We show that the sequence $\{u_k\}$ is bounded in $H^1(M)$.

Choose a small $\delta > 0$, such that the set

$$D = \{x \in M \mid R(x) > \delta\}$$

is non-empty. It is open since $R(x)$ is continuous. From the previous step, we know that $\{u_k\}$ is uniformly bounded in D.

Let $K(x)$ be a smooth function such that

$$K(x) < 0 \ , \ \text{for } x \in D; \quad \text{and } K(x) = 0 \ , \text{else where.}$$

For each α_k, let v_k be the unique solution of

$$-\triangle v_k + \alpha_k = K(x) e^{v_k} \ , \ \ x \in M.$$

Then since

$$-\triangle v_k + \alpha_k \leq 0 \ , \ \text{and } \alpha_k \rightarrow \alpha_o$$

The sequence $\{v_k\}$ is uniformly bounded.

Let $w_k = u_k - v_k$, then

$$-\triangle w_k = R(x) e^{u_k} - K(x) e^{v_k}. \tag{5.70}$$

We show that $\{w_k\}$ is bounded in $H^1(M)$.

On one hand, multiplying both sides of (5.70) by e^{w_k} and integrating on M, we have

$$\int_M |\bigtriangledown w_k|^2 e^{w_k} dA = \int_M R(x) e^{u_k + w_k} dA - \int_D K(x) e^{u_k} dA. \tag{5.71}$$

On the other hand, since each u_k is a minimizer of the functional J_k, the second derivative $J_k''(u_k)$ is positively definite. It follows that for any function $\phi \in H^1(M)$,

$$\int_M [|\bigtriangledown \phi|^2 - R(x) e^{u_k} \phi^2] d\,A \geq 0.$$

Choosing $\phi = e^{\frac{w_k}{2}}$, we derive

$$\frac{1}{4}\int_M |\bigtriangledown w_k|^2 e^{w_k} dA \geq \int_M R(x) e^{u_k+w_k} dA. \tag{5.72}$$

Combining (5.71) and (5.72), we arrive at

$$-\int_D K(x) e^{u_k} dA \geq \frac{3}{4}\int_M |\bigtriangledown w_k|^2 e^{w_k} dA. \tag{5.73}$$

Notice that $\{u_k\}$ is uniformly bounded in region D and $\{w_k\}$ is bounded from below due to the fact that $\{u_k\}$ is bounded from below and $\{v_k\}$ is bounded, (5.73) implies that $\int_M |\bigtriangledown w_k|^2 dA$ is bounded, and so does $\int_M |\bigtriangledown u_k|^2 dA$.

Consider $\tilde{u}_k = u_k - \bar{u}_k$, where $\bar{u}_k$ is the average of u_k. Obviously $\int_M \tilde{u}_k dA = 0$, hence we can apply Poincaré inequality to $\tilde{u}_k$ and conclude that $\{\tilde{u}_k\}$ is bounded in $H^1(M)$. Apparently, we have

$$\int_M u_k^2 dA \leq 2\left(\int_M \tilde{u}_k^2 dA + \bar{u}_k^2 |M|\right).$$

If $\int_M u_k^2 dA$ is unbounded, then $\{\bar{u}_k\}$ is unbounded. Taking into account that $\{u_k\}$ is bounded in the region D where $R(x)$ is positively bounded away from 0, we would have

$$\int_D \tilde{u}_k^2 dA = \int_D |u_k - \bar{u}_k|^2 dA \to \infty,$$

for some subsequence, a contradiction with the fact that $\{\tilde{u}_k\}$ is bounded in $H^1(M)$. Therefore, $\{u_k\}$ must also be bounded in $H^1(M)$. Consequently, there exists a subsequence of $\{u_k\}$ that converges to a function u_o in $H^1(M)$, and u_o is the desired weak solution for

$$-\triangle u_o + \alpha_o = R(x) e^{u_o}.$$

This completes the proof of the Theorem. □

6

Prescribing Scalar Curvature on S^n, for $n \geq 3$

6.1 Introduction

On higher dimensional sphere S^n with $n \geq 3$, Kazdan and Warner raise the following question:

Which functions $R(x)$ can be realized as the scalar curvature of some conformally related metrics? It is equivalent to consider the existence of a solution to the following semi-linear elliptic equation

$$-\triangle u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} R(x) u^{\frac{n+2}{n-2}}, \quad u > 0, \quad x \in S^n. \tag{6.1}$$

The equation is "critical" in the sense that lack of compactness occurs. Besides the obvious necessary condition that $R(x)$ be positive somewhere, there are well-known obstructions found by Kazdan and Warner [KW2] and later generalized by Bourguignon and Ezin [BE]. The conditions are:

$$\int_{S^n} X(R) dV_g = 0 \tag{6.2}$$

where dV_g is the volume element of the conformal metric g and X is any conformal vector field associated with the standard metric g_0. We call these Kazdan-Warner type conditions. These conditions give rise to many examples of $R(x)$ for which equation (6.1) has no solution. In particular, a monotone rotationally symmetric function R admits no solution.

In the last two decades, numerous studies were dedicated to these problems and various sufficient conditions were found (please see the articles [Ba] [BC] [C] [CC] [CD1] [CD2] [ChL] [CS] [CY1] [CY2] [CY3] [CY4] [ES] [Ha] [Ho] [Li] [Li2] [Li3] [Mo] [SZ] [XY] and the references therein). However, among others, one problem of common concern left open, namely, were those Kazdan-Warner type necessary conditions also sufficient ? In the case where R is rotationally symmetric, the conditions become:

$$R > 0 \text{ somewhere and } R' \text{ changes signs.} \tag{6.3}$$

Then
(a) is (6.3) a sufficient condition? and
(b) if not, what are the necessary and sufficient conditions?
Kazdan listed these as open problems in his CBMS Lecture Notes [Ka].

Recently, we answered question (a) negatively [CL3] [CL4]. We found some stronger obstructions. Our results imply that for a rotationally symmetric function R, if it is monotone in the region *where it is positive*, then problem (6.1) admits no solution unless R is a constant.

In other words, a necessary condition to solve (6.1) is that

$$R'(r) \text{ changes signs in the region } \textit{where } R \textit{ is positive.} \tag{6.4}$$

Now, is this a sufficient condition?

For prescribing Gaussian curvature equation (5.3) on S^2, Xu and Yang [XY] showed that if R is 'non-degenerate,' then (6.4) is a sufficient condition.

For equation (6.1) on higher dimensional spheres, a major difficulty is that multiple blow-ups may occur when approaching a solution by a minimax sequence of the functional or by solutions of subcritical equations.

In dimensions higher than 3, the 'non-degeneracy' condition is no longer sufficient to guarantee the existence of a solution. It was illustrated by a counter example in [Bi] constructed by Bianchi. He found some positive rotationally symmetric function R on S^4, which is non-degenerate and non-monotone, and for which the problem admits no solution. In this situation, a more proper condition is called the 'flatness condition'. Roughly speaking, it requires that at every critical point of R, the derivatives of R up to order $(n-2)$ vanish, while some higher but less than n^{th} order derivative is distinct from 0. For $n = 3$, the 'non-degeneracy' condition is a special case of the 'flatness condition'. Although people are wondering if the 'flatness condition' is necessary, it is still used widely today (see [CY3], [Li2], and [SZ]) as a standard assumption to guarantee the existence of a solution. The above mentioned Bianchi's counter example seems to suggest that the 'flatness condition' is somewhat sharp.

Now, a natural question is:

Under the 'flatness condition,' is (6.4) a sufficient condition?

In this section, we answer the question affirmatively. We prove that, under the 'flatness condition,' (6.4) is a necessary and sufficient condition for (6.1) to have a solution. This is true in all dimensions $n \geq 3$, and it applies to functions R with changing signs. Thus, we essentially answer the open question (b) posed by Kazdan.

There are many versions of 'flatness conditions,' a general one was presented in [Li2] by Y. Li. Here, to better illustrate the idea, in the statement of the following theorem, we only list a typical and easy-to-verify one.

Theorem 6.1.1 *Let $n \geq 3$. Let $R = R(r)$ be rotationally symmetric and satisfy the following flatness condition near every positive critical point r_o:*

$$R(r) = R(r_o) + a|r - r_o|^\alpha + h(|r - r_o|), \text{ with } a \neq 0 \text{ and } n-2 < \alpha < n. \tag{6.5}$$

where $h'(s) = o(s^{\alpha-1})$.

Then a necessary and sufficient condition for equation (6.1) to have a solution is that

$$R'(r) \text{ changes signs in the regions where } R > 0.$$

The theorem is proved by a variational approach. We blend in our new ideas with other ideas in [XY], [Ba], [Li2], and [SZ]. We use the 'center of mass' to define neighborhoods of 'critical points at infinity,' obtain some quite sharp and clean estimates in those neighborhoods, and construct a max-mini variational scheme at sub-critical levels and then approach the critical level.

Outline of the proof of Theorem 6.1.1.

Let $\gamma_n = \frac{n(n-2)}{4}$ and $\tau = \frac{n+2}{n-2}$. We first find a positive solution u_p of the subcritical equation

$$-\triangle u + \gamma_n u = R(r) u^p. \tag{6.6}$$

for each $p < \tau$ and close to τ. Then let $p \to \tau$, take the limit.

To find the solution of equation (6.6), we construct a max-mini variational scheme. Let

$$J_p(u) := \int_{S^n} R u^{p+1} dV$$

and

$$E(u) := \int_{S^n} (|\nabla u|^2 + \gamma_n u^2) dV.$$

We seek critical points of $J_p(u)$ under the constraint

$$H = \{u \in H^1(S^n) \mid E(u) = \gamma_n |S^n|, u \geq 0\}.$$

where $|S^n|$ is the volume of S^n.

If R has only one positive local maximum, then by condition (6.4), on each of the poles, R is either non-positive or has a local minimum. Then similar to the approach in [C], we seek a solution by minimizing the functional in a family of rotationally symmetric functions in H.

In the following, we assume that R has at least two positive local maxima. In this case, the solutions we seek are not necessarily symmetric.

Our scheme is based on the following key estimates on the values of the functional $J_p(u)$ in a neighborhood of the 'critical points at infinity' associated with each local maximum of R. Let r_1 be a positive local maximum of R. We prove that there is an open set $G_1 \subset H$ (independent of p), such that on the boundary ∂G_1 of G_1, we have

$$J_p(u) \leq R(r_1)|S^n| - \delta, \tag{6.7}$$

while there is a function $\psi_1 \in G_1$, such that

$$J_p(\psi_1) > R(r_1)|S^n| - \frac{\delta}{2}. \tag{6.8}$$

Here $\delta > 0$ is independent of p. Roughly speaking, we have some kinds of 'mountain pass' associated to each local maximum of R. The set G_1 is defined by using the 'center of mass'.

Let r_1 and r_2 be two smallest positive local maxima of R. Let ψ_1 and ψ_2 be two functions defined by (6.8) associated to r_1 and r_2. Let γ be a path in H connecting ψ_1 and ψ_2, and let Γ be the family of all such paths.

Define

$$c_p = \sup_{\gamma \in \Gamma} \min_{\gamma} J_p(u). \tag{6.9}$$

For each $p < \tau$, by compactness, there is a critical point u_p of $J_p(\cdot)$, such that

$$J_p(u_p) = c_p.$$

Obviously, a constant multiple of u_p is a solution of (6.6). Moreover, by (6.7),

$$J_p(u_p) \leq R(r_i)|S^n| - \delta, \tag{6.10}$$

for any positive local maximum r_i of R.

To find a solution of (6.1), we let $p \to \tau$, take the limit. To show the convergence of a subsequence of $\{u_p\}$, we established apriori bounds for the solutions in the following order.

(i) In the region where $R < 0$: This is done by the 'Kelvin Transform' and a maximum principle.

(ii) In the region where R is small: This is mainly due to the bounded-ness of the energy $E(u_p)$.

(iii) In the region where R is positively bounded away from 0: First due to the energy bound, $\{u_p\}$ can only blow up at most finitely many points. Using a Pohozaev type identity, we show that the sequence can only blow up at one point and the point must be a local maximum of R. Finally, we use (6.10) to argue that even one point blow up is impossible and thus establish an apriori bound for a subsequence of $\{u_p\}$.

In subsection 2, we carry on the max-mini variational scheme and obtain a solution u_p of (6.6) for each p.

In subsection 3, we establish a priori estimates on the solution sequence $\{u_p\}$.

6.2 The Variational Approach

In this section, we construct a max-mini variational scheme to find a solution of

$$-\triangle u + \gamma_n u = R(r)u^p \tag{6.11}$$

for each $p < \tau := \frac{n+2}{n-2}$.

Let

$$J_p(u) := \int_{S^n} R u^{p+1} dV$$

and

$$E(u) := \int_{S^n} (|\bigtriangledown u|^2 + \gamma_n u^2) dV.$$

Let

$$(u, v) = (\int_S \bigtriangledown u \bigtriangledown v + \gamma_n uv) dV$$

be the inner product in the Hilbert space $H^1(S^n)$, and

$$\|u\| := \sqrt{E(u)}$$

be the corresponding norm.

We seek critical points of $J_p(u)$ under the constraint

$$H := \{u \in H^1(S^n) \mid E(u) = E(1) = \gamma_n |S^n|, u \geq 0\}.$$

where $|S^n|$ is the volume of S^n.

One can easily see that a critical point of J_p in S multiplied by a constant is a solution of (6.11).

We divide the rest of the section into two parts.

First, we establish the key estimates (6.7) and (6.8).

Then, we carry on the max-mini variational scheme.

6.2.1 Estimate the Values of the Functional

To construct a max-mini variational scheme, we first show that there is some kinds of 'mountain pass' associated to each positive local maximum of R. Unlike the classical ones, these 'mountain passes' are in some neighborhoods of the 'critical points at infinity' (See Proposition 6.2.2 and 6.2.1 below).

Choose a coordinate system in R^{n+1}, so that the south pole of S^n is at the origin O and the center of the ball B^{n+1} is at $(0, \cdots, 0, 1)$. As usual, we use $|\cdot|$ to denote the distance in R^{n+1}.

Define the center of mass of u as

$$q(u) =: \frac{\int_{S^n} x u^{\tau+1}(x) dV}{\int_{S^n} u^{\tau+1}(x) dV} \tag{6.12}$$

It is actually the center of mass of the sphere S^n with density $u^{\tau+1}(x)$ at the point x.

We recall some well-known facts in conformal geometry.

Let ϕ_q be the standard solution with its 'center of mass' at $q \in B^{n+1}$, that is, $\phi_q \in H$ and satisfies

$$-\triangle u + \gamma_n u = \gamma_n u^{\tau}, \tag{6.13}$$

We may also regard ϕ_q as depend on two parameters λ and $\tilde{q}$, where $\tilde{q}$ is the intersection of S^n with the ray passing the center and the point q. When $\tilde{q} = O$ (the south pole of S^n), we can express, in the spherical polar coordinates $x = (r, \theta)$ of S^n $(0 \leq r \leq \pi, \theta \in S^{n-1})$ centered at the south pole where $r = 0$,

$$\phi_q = \phi_{\lambda,\tilde{q}} = \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}}\right)^{\frac{n-2}{2}}. \tag{6.14}$$

with $0 < \lambda \leq 1$. As $\lambda \to 0$, this family of functions blows up at south pole, while tends to zero elsewhere.

Correspondingly, there is a family of conformal transforms

$$T_q : H \longmapsto H; \quad T_q u := u(h_\lambda(x))[\det(dh_\lambda)]^{\frac{n-2}{2n}} \tag{6.15}$$

with

$$h_\lambda(r, \theta) = (2 \tan^{-1}(\lambda \tan \frac{r}{2}), \theta).$$

Here $\det(dh_\lambda)$ is the determinant of dh_λ, and can be expressed explicitly as

$$\det(dh_\lambda) = \left(\frac{\lambda}{\cos^2 \frac{r}{2} + \lambda^2 \sin^2 \frac{r}{2}}\right)^n.$$

It is well-known that this family of conformal transforms leave the equation (6.13), the energy $E(\cdot)$, and the integral $\int_{S^n} u^{\tau+1} dV$ invariant. More generally and more precisely, we have

Lemma 6.2.1 *For any $u, v \in H^1(S^n)$, and for any non-negative integer k, it holds the following*

$$(i) \ \ (T_q u, T_q v) = (u, v) \tag{6.16}$$

$$(ii) \ \ \int_{S^n} (T_q u)^k (T_q v)^{\tau+1-k} dV = \int_{S^n} u^k v^{\tau+1-k} dV. \tag{6.17}$$

In particular,

$$E(T_q u) = E(u) \ \ \text{and} \ \ \int_{S^n} (T_q u)^{\tau+1} dV = \int_{S^n} u^{\tau+1} dV.$$

Proof. (i) We will use a property of the conformal Laplacian to prove the invariance of the energy

$$E(T_q u) = E(u)\,.$$

The proof for (i) is similar.

On a n-dimensional Riemannian manifold (M, g),

$$L_g := \triangle_g - c(n) R_g$$

is call a conformal Laplacian, where $c(n) = \frac{n-2}{4(n-1)}$, $\triangle_g$ and R_g are the Laplace-Beltrami operator and the scalar curvature associated with the metric g respectively.

The conformal Laplacian has the following well-known invariance property under the conformal change of metrics. For $\hat{g} = w^{4/(n-2)} g$, $w > 0$, we have

$$L_{\hat{g}} u = w^{-\frac{n+2}{n-2}} L_g(uw)\,, \quad \text{for all } \ u \in C^\infty(M). \tag{6.18}$$

Interested readers may find its proof in [Bes].

If M is a compact manifold without boundary, then multiplying both sides of (6.18) by u and integrating by parts, one would arrive at

$$\int_M \{|\nabla_{\hat{g}} u|^2 + c(n) R_{\hat{g}} |u|^2\} dV_{\hat{g}} = \int_M \{|\nabla_g (uw)|^2 + c(n) R_g |uw|^2\} dV_g\,. \tag{6.19}$$

In our situation, we choose g to be the Euclidean metric $\bar{g}$ in R^n with $\bar{g}_{ij} = \delta_{ij}$ and $\hat{g} = g_o$, the standard metric on S^n. Then it is well-known that

$$g_o = w^{\frac{4}{n-2}}(x) \bar{g}$$

with

$$w(x) = \left(\frac{4}{4 + |x|^2} \right)^{\frac{n-2}{2}}$$

satisfying the equation

$$-\bar{\triangle} w = \gamma_n w^{\frac{n+2}{n-2}}.$$

We also have

$$c(n)R_{g_o} = \gamma_n := \frac{n(n-2)}{4}$$

and

$$c(n)R_{\bar{g}} = c(n)0 = 0.$$

Now formula (6.19) becomes

$$\int_{S^n} \{|\nabla_o u|^2 + \gamma_n |u|^2\} dV_o = \int_{R^n} |\bar{\nabla}(uw)|^2 dx, \tag{6.20}$$

where ∇_o and $\bar{\nabla}$ are gradient operators associated with the metrics g_o and $\bar{g}$ respectively.

Again, let $\tilde{q}$ be the intersection of S^n with the ray passing through the center of the sphere and the point q. Make a stereo-graphic projection from S^n to R^n as shown in the figure below, where $\tilde{q}$ is placed at the origin of R^n.

Under this projection, the point $(r, \theta) \in S^n$ becomes $x = (x_1, \cdots, x_n) \in R^n$, with

$$\frac{|x|}{2} = \tan \frac{r}{2}$$

and

$$(T_q u)(x) = u(\lambda x) \left(\frac{\lambda(4 + |x|^2)}{4 + \lambda |x|^2} \right)^{\frac{n-2}{2}}.$$

It follows from this and (6.20) that

$$\begin{aligned}
E(T_q u) &= \int_{R^n} \left\{ |\nabla_o \left[u(\lambda x) \left(\frac{\lambda(4 + |x|^2)}{4 + \lambda |x|^2} \right)^{\frac{n-2}{2}} \right] |^2 + \gamma_n |u(\lambda x) \left(\frac{\lambda(4 + |x|^2)}{4 + \lambda |x|^2} \right)^{\frac{n-2}{2}} |^2 \right\} dV_{g_o} \\
&= \int_{R^n} |\bar{\nabla} \left[u(\lambda x) \left(\frac{\lambda(4 + |x|^2)}{4 + \lambda |x|^2} \right)^{\frac{n-2}{2}} w(x) \right] |^2 dx \\
&= \int_{R^n} |\bar{\nabla} \left[u(\lambda x) \left(\frac{4\lambda}{4 + \lambda |x|^2} \right)^{\frac{n-2}{2}} \right] |^2 dx \\
&= \int_{R^n} |\bar{\nabla} \left[u(y) \left(\frac{4}{4 + |y|^2} \right)^{\frac{n-2}{2}} \right] |^2 dy \\
&= \int_{R^n} |\bar{\nabla}(u(y)w(y))|^2 dy \\
&= E(u).
\end{aligned}$$

(ii) This can be derived immediately by making change of variable $y = h_\lambda(x)$.

This completes the proof of the lemma. □

One can also verify that

$$T_q \phi_q = 1.$$

The relations between q and λ, $\tilde{q}$ and T_q for $\tilde{q} \neq O$ can be expressed in a similar way.

We now carry on the estimates near the south pole $(0, \theta)$ which we assume to be a positive local maximum. The estimates near other positive local maxima are similar. Our conditions on R implies that

$$R(r) = R(0) - ar^{\alpha}, \text{ for some } a > 0,\ n-2 < \alpha < n. \tag{6.21}$$

in a small neighborhood of O.

Define

$$\Sigma = \{u \in H \mid |q(u)| \leq \rho_o, \|v\| := \min_{t,q} \|u - t\phi_q\| \leq \rho_o\}. \tag{6.22}$$

Notice that the 'centers of mass' of functions in Σ are near the south pole O. This is a neighborhood of the 'critical points at infinity' corresponding to O. We will estimate the functional J_p in this neighborhood.

We first notice that the supremum of J_p in Σ approaches $R(0)|S^n|$ as $p{\rightarrow}\tau$. More precisely, we have

Proposition 6.2.1 *For any $\delta_1 > 0$, there is a $p_1 \leq \tau$, such that for all $\tau \geq p \geq p_1$,*

$$\sup_{\Sigma} J_p(u) > R(0)|S^n| - \delta_1. \tag{6.23}$$

Then we show that on the boundary of Σ, J_p is strictly less and bounded away from $R(0)|S^n|$.

Proposition 6.2.2 *There exist positive constants ρ_o, p_o, and δ_o, such that for all $p \geq p_o$ and for all $u \in \partial\Sigma$, holds*

$$J_p(u) \leq R(0)|S^n| - \delta_o. \tag{6.24}$$

Proof of Proposition 6.2.1

Through a straight forward calculation, one can show that

$$J_\tau(\phi_{\lambda,O}){\rightarrow}R(0)|S^n|, \quad \text{as } \lambda{\rightarrow}0. \tag{6.25}$$

For a given $\delta_1 > 0$, by (6.25), one can choose λ_o, such that $\phi_{\lambda_o,O} \in \Sigma$, and

$$J_\tau(\phi_{\lambda_o,O}) > R(0)|S^n| - \frac{\delta_1}{2}. \tag{6.26}$$

It is easy to see that for a fixed function $\phi_{\lambda_o,O}$, J_p is continuous with respect to p. Hence, by (6.26), there exists a p_1, such that for all $p \geq p_1$,

$$J_p(\phi_{\lambda_o,O}) > R(0)|S^n| - \delta_1.$$

This completes the proof of Proposition 6.2.1.

The proof of Proposition 6.2.2 is rather complex. We first estimate J_p for a family of standard functions in Σ.

Lemma 6.2.2 *For p sufficiently close to τ, and for λ and $|\tilde{q}|$ sufficiently small, we have*

$$J_p(\phi_{\lambda,\tilde{q}}) \leq (R(0) - C_1|\tilde{q}|^\alpha)|S^n|(1 + o_p(1)) - C_1\lambda^{\alpha+\delta_p}, \tag{6.27}$$

where $\delta_p := \tau - p$ and $o_p(1) \to 0$ as $p \to \tau$.

Proof of Lemma 6.2.2. Let $\epsilon > 0$ be small such that (6.21) holds in $B_\epsilon(O) \subset S^n$. We express

$$J_p(\phi_{\lambda,\tilde{q}}) = \int_{S^n} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV = \int_{B_\epsilon(O)} \cdots + \int_{S^n \setminus B_\epsilon(O)} \cdots. \tag{6.28}$$

From the definition (6.14) of $\phi_{\lambda,\tilde{q}}$ and the bounded-ness of R, we obtain the smallness of the second integral:

$$\int_{S^n \setminus B_\epsilon(O)} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV \leq C_2\lambda^{n-\delta_p\frac{n-2}{2}} \quad \text{for } \tilde{q} \in B_{\frac{\epsilon}{2}}(O). \tag{6.29}$$

To estimate the first integral, we work in a local coordinate centered at O.

$$\begin{aligned}
\int_{B_\epsilon(O)} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV &= \int_{B_\epsilon(O)} R(x+\tilde{q})\phi_{\lambda,O}^{p+1} dV \\
&\leq \int_{B_\epsilon(O)} [R(0) - a|x+\tilde{q}|^\alpha]\phi_{\lambda,O}^{p+1} dV \\
&\leq \int_{B_\epsilon(O)} [R(0) - C_3|x|^\alpha - C_3|\tilde{q}|^\alpha]\phi_{\lambda,O}^{p+1} dV \\
&\leq [R(0) - C_3|\tilde{q}|^\alpha]|S^n|[1 + o_p(1)] - C_4\lambda^{\alpha+\delta_p(n-2)/2}.
\end{aligned} \tag{6.30}$$

Here we have used the fact that $|x+\tilde{q}|^\alpha \geq c(|x|^\alpha + |\tilde{q}|^\alpha)$ for some $c > 0$ in one half of the ball $B_\epsilon(O)$ and the symmetry of $\phi_{\lambda,O}$.

Noticing that $\alpha < n$ and $\delta_p \to 0$, we conclude that (6.28), (6.29), and (6.30) imply (6.27). This completes the proof of the Lemma.

To estimate J_p for all $u \in \partial\Sigma$, we also need the following two lemmas that describe some useful properties of the set Σ.

Lemma 6.2.3 (On the 'center of mass')

(i) Let q, λ, and $\tilde{q}$ be defined by (6.14). Then for sufficiently small q,

$$|q|^2 \leq C(|\tilde{q}|^2 + \lambda^4). \tag{6.31}$$

(ii) Let ρ_o, q, and v be defined by (6.22). Then for ρ_o sufficiently small,

$$\rho_o \leq |q| + C\|v\|. \tag{6.32}$$

Lemma 6.2.4 *(Orthogonality)*

Let $u \in \Sigma$ and $v = u - t_o \phi_{q_o}$ be defined by (6.22). Let T_{q_o} be the conformal transform such that

$$T_{q_o} \phi_{q_o} = 1. \tag{6.33}$$

Then

$$\int_{S^n} T_{q_o} v dV = 0 \tag{6.34}$$

and

$$\int_{S^n} T_{q_o} v \cdot \psi_i(x) dV = 0, \quad i = 1, 2, \cdots n. \tag{6.35}$$

where ψ_i are first spherical harmonic functions on S^n:

$$-\triangle \psi_i = n\, \psi_i(x)\, , i = 1, 2, \cdots, n.$$

If we place the center of the sphere S^n at the origin of R^{n+1} (not in our case), then for $x = (x_1, \cdots, x_n)$,

$$\psi_i(x) = x_i\, , i = 1, 2, \cdots, n.$$

Proof of Lemma 6.2.3.

(i) From the definition that $\phi_q = \phi_{\lambda, \tilde{q}}$, and (6.14), and by an elementary calculation, one arrives at

$$|q - \tilde{q}| \sim \lambda^2, \quad \text{for } \lambda \text{ sufficiently small} . \tag{6.36}$$

Draw a perpendicular line segment from O to $\overline{q\tilde{q}}$, then one can see that (6.31) is a direct consequence of the triangle inequality and (6.36).

(ii) For any $u = t\phi_q + v \in \partial\Sigma$, the boundary of Σ, by the definition, we have either $\|v\| = \rho_o$ or $|q(u)| = \rho_o$. If $\|v\| = \rho_o$, then we are done. Hence we assume that $|q(u)| = \rho_o$. It follows from the definition of the 'center of mass' that

$$\rho_o = |\frac{\int_{S^n} x(t\phi_q + v)^{\tau+1}}{\int_{S^n} (t\phi_q + v)^{\tau+1}}|. \tag{6.37}$$

Noticing that $\|v\| \leq \rho_o$ is very small, we can expand the integrals in (6.37) in terms of $\|v\|$:

$$\begin{aligned} \rho_o &= |\frac{t^{\tau+1} \int x\phi_q^{\tau+1} + (\tau+1)t^\tau \int x\phi_q^\tau v + o(\|v\|)}{t^{\tau+1} \int \phi_q^{\tau+1} + (\tau+1)t^\tau \int \phi_q^\tau v + o(\|v\|)}| \\ &\leq |\frac{\int x\phi_q^{\tau+1}}{\int \phi_q^{\tau+1}}| + C\|v\| \leq |q| + C\|v\|. \end{aligned}$$

Here we have used the fact that v is small and that t is close to 1. This completes the proof of Lemma 6.2.3.

Proof of Lemma 6.2.4.

Write $L = \triangle + \gamma_n$. We use the fact that $v = u - t_o\phi_{q_o}$ is the minimum of $E(u - t\phi_q)$ among all possible values of t and q.

(i) By a variation with respect to t, we have

$$0 = (v, \phi_{q_o}) = \int_{S^n} vL\phi_{q_o} dV. \tag{6.38}$$

It follows from (6.13) that

$$\int_{S^n} v\phi_{q_o}^{\tau} dV = 0.$$

Now, apply the conformal transform T_{q_o} to both v and ϕ_{q_o} in the above integral. By the invariant property of the transform, we arrive at (6.34).

(ii) To prove (6.35), we make a variation with respect to q.

$$0 = \bigtriangledown_q \int vL\phi_q \mid_{q=q_o} = \gamma_n \bigtriangledown_q \int v\phi_q^{\tau} \mid_{q=q_o}$$

$$\bigtriangledown_q \int T_{q_o} v (T_{q_o}\phi_q)^{\tau} \mid_{q=q_o} = \bigtriangledown_q \int T_{q_o} v (\phi_{q-q_o})^{\tau} \mid_{q=q_o}$$

$$\tau \int T_{q_o} v \phi_{q-q_o}^{\tau-1} \bigtriangledown_q \phi_{q-q_o} \mid_{q=q_o}$$

$$= \tau \int T_{q_o} v (\psi_1, \cdots, \psi_n).$$

This completes the proof of the Lemma.

Proof of Proposition 6.2.2

Make a perturbation of $R(x)$:

$$\bar{R}(x) = \begin{cases} R(x) & x \in B_{2\rho_o}(O) \\ m & x \in S^n \setminus B_{2\rho_o}(O), \end{cases} \tag{6.39}$$

where $m = R \mid_{\partial B_{2\rho_o}(O)}$.

Define

$$\bar{J}_p(u) = \int_{S^n} \bar{R}(x) u^{p+1} dV.$$

The estimate is divided into two steps.

In step 1, we show that the difference between $J_p(u)$ and $\bar{J}_\tau(u)$ is very small. In step 2, we estimate $\bar{J}_\tau(u)$.

Step 1.

First we show

$$\bar{J}_p(u) \leq \bar{J}_\tau(u)(1 + o_p(1)). \tag{6.40}$$

where $o_p(1) \rightarrow 0$ as $p \rightarrow \tau$.

In fact, by the Hölder inequality

$$\int \bar{R}u^{p+1} \leq \left(\int \bar{R}^{\frac{\tau+1}{p+1}} u^{\tau+1} dV \right)^{\frac{p+1}{\tau+1}} |S^n|^{\frac{\tau-p}{\tau+1}}$$

$$\leq \left(\int \bar{R}u^{\tau+1} dV \right)^{\frac{p+1}{\tau+1}} \mid R(0)|S^n| \mid^{\frac{\tau-p}{\tau+1}} .$$

This implies (6.40).

Now estimate the difference between $J_p(u)$ and $\bar{J}_p(u)$.

$$\mid J_p(u) - \bar{J}_p(u) \mid \leq C_1 \int_{S^n \setminus B_{2\rho_o}(O)} u^{p+1}$$

$$\leq C_2 \int_{S^n \setminus B_{2\rho_o}(O)} (t\phi_q)^{p+1} + C_2 \int_{S^n \setminus B_{2\rho_o}(O)} v^{p+1} \leq C_3 \lambda^{n-\delta_p \frac{n-2}{2}} + C_3 \|v\|^{p+1}. \tag{6.41}$$

Step 2.

By virtue of (6.40) and (6.41), we now only need to estimate $\bar{J}_\tau(u)$. For any $u \in \partial\Sigma$, write $u = v + t\phi_q$. (6.38) implies that v and ϕ_q are orthogonal with respect to the inner product associated to $E(\cdot)$, that is

$$\int_{S^n} (\bigtriangledown v \bigtriangledown \phi_q + \gamma_n v \phi_q) dV = 0.$$

Notice that $\|u\|^2 := E(u)$, we have

$$\|u\|^2 = \|v\|^2 + t^2 \|\phi_q\|^2.$$

Using the fact that both u and ϕ_q belong to H with

$$\|u\| = \gamma_n |S^n| = \|\phi_q\|,$$

We derive

$$t^2 = 1 - \frac{\|v\|^2}{\gamma_n |S^n|}. \tag{6.42}$$

It follows that

$$\int_{S^n} \bar{R}(x) u^{\tau+1} \leq$$

$$t^{\tau+1} \int_{S^n} \bar{R}(x) \phi_q^{\tau+1} + (\tau+1) \int_{S^n} \bar{R}(x) \phi_q^{\tau} v + \frac{\tau(\tau+1)}{2} \int_{S^n} \phi_q^{\tau-1} v^2 + o(\|v\|^2)$$

$$= I_1 + (\tau+1) I_2 + \frac{\tau(\tau+1)}{2} I_3 + o(\|v\|^2). \tag{6.43}$$

To estimate I_1, we use (6.42) and (6.27),

$$I_1 \leq (1 - \frac{\tau+1}{2}\frac{\|v\|^2}{\gamma_n |S^n|}) R(0)|S^n|(1 - c_1|\tilde{q}|^\alpha - c_1\lambda^\alpha) + O(\lambda^n) + o(\|v\|^2). \quad (6.44)$$

for some positive constant c_1.

To estimate I_2, we use the H^1 orthogonality between v and ϕ_q (see (6.38)),

$$\begin{aligned} I_2 &= \int_{S^n} (\bar{R}(x) - m)\phi_q^\tau v dV = \int_{B_{2\rho_o}(O)} (\bar{R}(x) - m)\phi_q^\tau v dV \\ &= \frac{1}{\gamma_n} \int_{B_{2\rho_o}(O)} (\bar{R}(x) - m) L\phi_q \cdot v dV \\ &\leq C\rho_o^\alpha \mid \int_{B_{2\rho_o}(O)} L\phi_q \cdot v dV \mid = C\rho_o^\alpha |(\phi_q, v)| \\ &\leq C\rho_o^\alpha \|\phi_q\| \|v\| \leq C_4 \rho_o^\alpha \|v\|. \end{aligned} \quad (6.45)$$

To estimate I_3, we use (6.34) and (6.35). It is well-known that the first and second eigenspaces of $-\triangle$ on S^n are constants and ψ_i corresponding to eigenvalues 0 and n respectively. Now (6.34) and (6.35) imply that $T_q v$ is orthogonal to these two eigenspaces and hence

$$\int_{S^n} \triangledown(T_q v)|^2 dV \geq \lambda_2 \int_{S^n} |T_q v|^2 dV \,,$$

where $\lambda_2 = n + c_2$, for some positive number c_2, is the second non-zero eigenvalue of $-\triangle$. Consequently

$$\|v\|^2 = \|T_q v\|^2 \geq (\gamma_n + n + c_2) \int_{S^n} (T_q v)^2.$$

It follows that

$$\begin{aligned} I_3 &\leq R(0) \int_{S^n} \phi_q^{\tau-1} v^2 dV = R(0) \int_{S^n} (T_q\phi_q)^{\tau-1} (T_q v)^2 dV \\ &= R(0) \int_{S^n} (T_q v)^2 \leq \frac{R(0)}{\gamma_n + n + c_2} \|v\|^2. \end{aligned} \quad (6.46)$$

Now (6.44), (6.45), and (6.46) imply that there is a $\beta > 0$ such that

$$\bar{J}_\tau(u) \leq R(0)|S^n|[1 - \beta(|\tilde{q}|^\alpha + \lambda^\alpha + \|v\|^2)]. \quad (6.47)$$

Here we have used the fact that $\|v\|$ is very small and $\rho_o^\alpha \|v\|$ can be controlled by $|\tilde{q}|^\alpha + \lambda^\alpha$ (see Lemma 6.2.2). Notice that the positive constant c_2 in (6.46) has played a key role, and without it, the coefficient of $\|v\|^2$ in (6.47) would be 0 due to the fact that $\gamma_n = \frac{n(n-1)}{4}$.

For any $u \in \partial\Sigma$, we have

$$\text{either } \; \|v\| = \rho_o \; \text{ or } |q(u)| = \rho_o.$$

By (6.31) and (6.47), in either case there exist a $\delta_o > 0$, such that

$$\bar{J}_\tau(u) \le R(0)|S^n| - 2\delta_o. \tag{6.48}$$

Now (6.24) is an immediate consequence of (6.40), (6.41), and (6.48) . This completes the proof of the Proposition.

6.2.2 The Variational Scheme

In this part, we construct variational schemes to show the existence of a solution to equation (6.11) for each $p < \tau$.

Case (i): R has only one positive local maximum.

In this case, condition (6.4) implies that R must have local minima at both poles. Similar to the ideas in [C], we seek a solution by maximizing the functional J_p in a class of rotationally symmetric functions:

$$H_r = \{u \in H \mid u = u(r)\}. \tag{6.49}$$

Obviously, J_p is bounded from above in H_r and it is well-known that the variational scheme is compact for each $p < \tau$ (See the argument in Chapter 2). Therefore any maximizing sequence possesses a converging subsequence in H_r and the limiting function multiplied by a suitable constant is a solution of (6.11).

Case (ii): R has at least two positive local maxima.

Let r_1 and r_2 be the two smallest positive local maxima of R. By Proposition 6.2.1 and 6.2.2, there exist two disjoint open sets $\Sigma_1, \Sigma_2 \subset H, \psi_i \in \Sigma_i, p_o < \tau$ and $\delta > 0$, such that for all $p \ge p_o$,

$$J_p(\psi_i) > R(r_i)|S^n| - \frac{\delta}{2},\ i = 1, 2; \tag{6.50}$$

and

$$J_p(u) \le R(r_i)|S^n| - \delta,\ \ \forall u \in \partial\Sigma_i,\ i = 1, 2. \tag{6.51}$$

Let γ be a path in H joining ψ_1 and ψ_2. Let Γ be the family of all such paths. Define

$$c_p = \sup_{\gamma \in \Gamma} \min_{u \in \gamma} J_p(u). \tag{6.52}$$

For each fixed $p < \tau$, by the well-known compactness, there exists a critical (saddle) point u_p of J_p with

$$J_p(u_p) = c_p.$$

Moreover, due to (6.51) and the definition of c_p, we have

$$J_p(u_p) \le \min_{i=1,2} R(r_i)|S^n| - \delta. \tag{6.53}$$

One can easily see that a critical point of J_p in H multiplied by a constant is a solution of (6.11) and for all p close to τ, the constant multiples are bounded from above and bounded away from 0.

6.3 The A Priori Estimates

In the previous section, we showed the existence of a positive solution u_p to the subcritical equation (6.11) for each $p < \tau$. In this section, we prove that as $p{\rightarrow}\tau$, there is a subsequence of $\{u_p\}$, which converges to a solution u_o of (6.1). The convergence is based on the following a priori estimate.

Theorem 6.3.1 *Assume that R satisfies condition (6.5) in Theorem 6.1.1, then there exists $p_o < \tau$, such that for all $p_o < p < \tau$, the solution of (6.11) obtained by the variational scheme are uniformly bounded.*

To prove the Theorem, we estimate the solutions in three regions where $R < 0, R$ close to 0, and $R > 0$ respectively.

6.3.1 In the Region Where $R < 0$

The apriori bound of the solutions is stated in the following proposition, which is a direct consequence of Proposition 6.2.2 in our paper [CL6].

Proposition 6.3.1 *For all $1 < p \leq \tau$, the solutions of (6.11) are uniformly bounded in the regions where $R \leq -\delta$, for every $\delta > 0$. The bound depends only on δ, dist$(\{x \mid R(x) \leq -\delta\}, \{x \mid R(x) = 0\})$, and the lower bound of R.*

6.3.2 In the Region Where R is Small

In [CL6], we also obtained estimates in this region by the method of moving planes. However, in that paper, we assume that $\bigtriangledown R$ be bounded away from zero. In [CL7], for rotationally symmetric functions R on S^3, we removed this condition by using a blowing up analysis near $R(x) = 0$. That method may be applied to higher dimensions with a few modifications. Now within our variational frame in this paper, the a priori estimate is simpler. It is mainly due to the energy bound.

Proposition 6.3.2 *Let $\{u_p\}$ be solutions of equation (6.11) obtained by the variational approach in section 2. Then there exists a $p_o < \tau$ and a $\delta > 0$, such that for all $p_o < p \leq \tau$, $\{u_p\}$ are uniformly bounded in the regions where $|R(x)| \leq \delta$.*

Proof. It is easy to see that the energy $E(u_p)$ are bounded for all p. Now suppose that the conclusion of the Proposition is violated. Then there exists a subsequence $\{u_i\}$ with $u_i = u_{p_i}, p_i{\rightarrow}\tau$, and a sequence of points $\{x^i\}$, with $x^i{\rightarrow}x^o$ and $R(x^o) = 0$, such that

$$u_i(x^i){\rightarrow}\infty.$$

We will use a re-scaling argument to derive a contradiction. Since x^i may not be a local maximum of u_i, we need to choose a point near x^i, which is almost a local maximum. To this end, let K be any large number and let

$$r_i = 2K[u_i(x^i)]^{-\frac{p_i-1}{2}}.$$

In a small neighborhood of x^o, choose a local coordinate and let

$$s_i(x^i) = u_i(x)(r_i - |x - x^i|)^{\frac{2}{p_i-1}}.$$

Let a_i be the maximum of $s_i(x)$ in $B_{r_i}(x^i)$. Let $\lambda_i = [u_i(a_i)]^{-\frac{p_i-1}{2}}$.

Then from the definition of a_i, one can easily verify that the ball $B_{\lambda_i K}(a_i)$ is in the interior of $B_{r_i}(x^i)$ and the value of $u_i(x)$ in $B_{\lambda_i K}(a_i)$ is bounded above by a constant multiple of $u_i(a_i)$.

To see the first part, we use the fact $s_i(a_i) \geq s_i(x^i)$. From this, we derive

$$u_i(a_i)(r_i - |a_i - x^i|)^{\frac{2}{p_i-1}} \geq u_i(x^i) r_i^{\frac{2}{p_i-1}} = (2K)^{\frac{2}{p_i-1}}.$$

It follows that

$$\frac{r_i - |a_i - x^i|}{2} \geq \lambda_i K, \tag{6.54}$$

hence

$$B_{\lambda_i K}(a_i) \subset B_{r_i}(x^i).$$

To see the second part, we use $s_i(a_i) \geq s_i(x)$ and derive from (6.54),

$$\begin{aligned} u_i(x) &\leq u_i(a_i) \left(\frac{r_i - |a_i - x^i|}{r_i - |x - x^i|} \right)^{\frac{2}{p_i-1}} \\ &\leq u_i(a_i)\, 2^{\frac{2}{p_i-1}} \quad , \qquad \forall\, x \in B_{\lambda_i K}(a_i). \end{aligned}$$

Now we can make a rescaling.

$$v_i(x) = \frac{1}{u_i(a_i)} u_i(\lambda_i x + a_i).$$

Obviously v_i is bounded in $B_K(0)$, and it follows by a standard argument that $\{v_i\}$ converges to a harmonic function v_o in $B_K(0) \subset R^n$, with $v_o(0) = 1$. Consequently for i sufficiently large,

$$\int_{B_K(0)} v_i(x)^{\tau+1} dx \geq cK^n, \tag{6.55}$$

for some $c > 0$.

On the other hand, the bounded-ness of the energy $E(u_i)$ implies that

$$\int_{S^n} u_i^{\tau+1} dV \leq C, \tag{6.56}$$

for some constant C. By a straight forward calculation, one can verify that, for any $K > 0$,

$$\int_{S^n} u_i^{\tau+1} dV \geq \int_{B_K(0)} v_i^{\tau+1} dx. \tag{6.57}$$

Obviously, (6.56) and (6.57) contradict with (6.55). This completes the proof of the Proposition. □

6.3.3 In the Regions Where $R > 0$.

Proposition 6.3.3 *Let $\{u_p\}$ be solutions of equation (6.11) obtained by the variational approach in section 2. Then there exists a $p_o < \tau$, such that for all $p_o < p \leq \tau$ and for any $\epsilon > 0$, $\{u_p\}$ are uniformly bounded in the regions where $R(x) \geq \epsilon$.*

Proof. The proof is divided into 3 steps. In Step 1, we argue that the solutions can only blow up at finitely many points because of the energy bound. In Step 2, we show that there is no more than one point blow up and the point must be a local maximum of R. This is done mainly by using a Pohozaeve type identity. In Step 3, we use (6.53) to conclude that even one point blow up is impossible.

Step 1. The argument is standard. Let $\{x^i\}$ be a sequence of points such that $u_i(x^i) \rightarrow \infty$ and $x^i \rightarrow x_o$ with $R(x_o) > 0$. Let r_i, $s_i(x)$, and $v_i(x)$ be defined as in Part II. Then similar to the argument in Part II, we see that $\{v_i(x)\}$ converges to a standard function $v_o(x)$ in R^n with

$$-\triangle v_o = R(x_o) v_o^\tau.$$

It follows that

$$\int_{B_{r_i}(x_o)} u_i^{\tau+1} dV \geq c_o > 0.$$

Because the total energy of u_i is bounded, we can have only finitely many such x_o. Therefore, $\{u_i\}$ can only blow up at finitely many points.

Step 2. We have shown that $\{u_i\}$ has only isolated blow up points. As a consequence of a result in [Li2] (also see [CL6] or [SZ]), we have

Lemma 6.3.1 *Let R satisfy the 'flatness condition' in Theorem 1, Then $\{u_i\}$ can have at most one simple blow up point and this point must be a local maximum of R. Moreover, $\{u_i\}$ behaves almost like a family of the standard functions $\phi_{\lambda_i, \tilde{q}}$, with $\lambda_i = (\max u_i)^{-\frac{2}{n-2}}$ and with $\tilde{q}$ at the maximum of R.*

Step 3. Finally, we show that even one point blow up is impossible. For convenience, let $\{u_i\}$ be the sequence of critical points of J_{p_i} we obtained in Section 2. From the proof of Proposition 2.2, one can obtain

$$J_\tau(u_i) \leq \min_k R(r_k)|S^n| - \delta, \tag{6.58}$$

for all positive local maxima r_k of R, because each u_i is the minimum of J_{p_i} on a path. Now if $\{u_i\}$ blow up at x_o, then by Lemma 6.3.1, we have

$$J_\tau(u_i) \rightarrow R(x_o)|S^n|.$$

This is a contradiction with (6.58).

Therefore, the sequence $\{u_i\}$ is uniformly bounded and possesses a subsequence converging to a solution of (6.1). This completes the proof of Theorem 6.1.1. □

7

Maximum Principles

7.1 Introduction

As a simple example of maximum principle, let's consider a C^2 function $u(x)$ of one independent variable x. It is well-known in calculus that at a local maximum point x_o of u, we must have

$$u''(x_o) \leq 0.$$

Based on this observation, then we have the following simplest version of maximum principle:

Assume that $u''(x) > 0$ in an open interval (a, b), then u can not have any interior maximum in the interval.

One can also see this geometrically. Since under the condition $u''(x) > 0$, the graph is concave up, it can not have any local maximum.

More generally, for a C^2 function $u(x)$ of n-independent variables $x = (x_1, \cdots, x_n)$, at a local maximum x^o, we have

$$D^2u(x^o) := (u_{x_i x_j}(x^o)) \leq 0\,,$$

that is, the symmetric matrix is non-positive definite at point x^o. Correspondingly, the simplest version maximum principle reads:

If

$$(a_{ij}(x)) \geq 0 \quad and \; \sum_{ij} a_{ij}(x) u_{x_i x_j}(x) > 0 \tag{7.1}$$

in an open bounded domain Ω, then u can not achieve its maximum in the interior of the domain.

An interesting special case is when $(a_{ij}(x))$ is an identity matrix, in which condition (7.1) becomes

$$\triangle u > 0.$$

Unlike its one-dimensional counterpart, condition (7.1) no longer implies that the graph of $u(x)$ is concave up. A simple counter example is

$$u(x_1, x_2) = x_1^2 - \frac{1}{2} x_2^2.$$

One can easily see from the graph of this function that $(0,0)$ is a saddle point.

In this case, the validity of the maximum principle comes from the simple algebraic fact:

For any two $n \times n$ matrices A and B, if $A \geq 0$ and $B \leq 0$, then $AB \leq 0$.

In this chapter, we will introduce various maximum principles, and most of them will be used in the method of moving planes in the next chapter. Besides this, there are numerous other applications. We will list some below.

i) *Providing Estimates*

Consider the boundary value problem

$$\begin{cases} -\triangle u = f(x) \,, \; x \in B_1(0) \subset R^n \\ u(x) = 0 \,, \qquad x \in \partial B_1(0). \end{cases} \tag{7.2}$$

If $a \leq f(x) \leq b$ in $B_1(0)$, then we can compare the solution u with the two functions

$$\frac{a}{2n}(1 - |x|^2) \;\text{ and }\; \frac{b}{2n}(1 - |x|^2)$$

which satisfy the equation with $f(x)$ replaced by a and b, respectively, and which vanish on the boundary as u does. Now, applying the maximum principle for $\triangle$ operator (see Theorem 7.1.1 in the following), we obtain

$$\frac{a}{2n}(1 - |x|^2) \leq u(x) \leq \frac{b}{2n}(1 - |x|^2).$$

ii) *Proving Uniqueness of Solutions*

In the above example, if $f(x) \equiv 0$, then we can choose $a = b = 0$, and this implies that $u \equiv 0$. In other words, the solution of the boundary value problem (7.2) is unique.

iii) *Establishing the Existence of Solutions*

(a) For a linear equation such as (7.2) in any bounded open domain Ω, let

$$u(x) = \sup \phi(x)$$

where the sup is taken among all the functions that satisfy the corresponding differential inequality

$$\begin{cases} -\triangle\phi \leq f(x)\ ,\ x \in \Omega \\ \phi(x) = 0\ , \qquad x \in \partial\Omega. \end{cases}$$

Then u is a solution of (7.2).

(b) Now consider the nonlinear problem

$$\begin{cases} -\triangle u = f(u)\ , x \in \Omega \\ u(x) = 0\ , x \in \partial\Omega. \end{cases} \tag{7.3}$$

Assume that $f(\cdot)$ is a smooth function with $f'(\cdot) \geq 0$. Suppose that there exist two functions $\underline{u}(x) \leq \bar{u}(x)$, such that

$$-\triangle\underline{u} \leq f(\underline{u}) \leq f(\bar{u}) \leq -\triangle\bar{u}.$$

These two functions are called sub (or lower) and super (or upper) solutions respectively.

To seek a solution of problem (7.3), we use successive approximations. Let

$$-\triangle u_1 = f(\underline{u}) \quad \text{and} \quad -\triangle u_{i+1} = f(u_i).$$

Then by maximum principle, we have

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq \cdots \leq \bar{u}.$$

Let u be the limit of the sequence $\{u_i\}$:

$$u(x) = \lim u_i(x),$$

then u is a solution of the problem (7.3).

In Section 7.2 , we introduce and prove the weak maximum principles.

Theorem 7.1.1 *(Weak Maximum Principle for $-\triangle$.)*

i) If

$$-\triangle u(x) \geq 0, \quad x \in \Omega,$$

then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u.$$

ii) If

$$-\triangle u(x) \leq 0, \quad x \in \Omega,$$

then

$$\max_{\bar{\Omega}}\ u \leq \max_{\partial\Omega}\ u.$$

This result can be extended to general uniformly elliptic operators. Let

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}.$$

Define

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x).$$

Here we always assume that $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are bounded continuous functions in $\bar{\Omega}$. We say that L is uniformly elliptic if

$$a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \text{for any } x \in \Omega, \quad \text{any } \xi \in R^n \text{ and for some } \delta > 0.$$

Theorem 7.1.2 *(Weak Maximum Principle for L) Let L be the uniformly elliptic operator defined above. Assume that $c(x) \equiv 0$.*

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u.$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} .$$

These Weak Maximum Principles infer that the minima or maxima of u attain at some points on the boundary $\partial\Omega$. However, they do not exclude the possibility that the minima or maxima may also occur in the interior of Ω. Actually this can not happen unless u is constant, as we will see in the following.

Theorem 7.1.3 *(Strong Maximum Principle for L with $c(x) \equiv 0$) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \equiv 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u attains its minimum value only on $\partial\Omega$ unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u attains its maximum value only on $\partial\Omega$ unless u is constant.

This maximum principle (as well as the weak one) can also be applied to the case when $c(x) \geq 0$ with slight modifications.

Theorem 7.1.4 *(Strong Maximum Principle for L with $c(x) \geq 0$) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \geq 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u can not attain its non-positive minimum in the interior of Ω unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u can not attain its non-negative maximum in the interior of Ω unless u is constant.

We will prove these Theorems in Section 7.3 by using the Hopf Lemma.

Notice that in the previous Theorems, we all require that $c(x) \geq 0$. Roughly speaking, maximum principles hold for 'positive' operators. $-\triangle$ is 'positive', and obviously so does $-\triangle + c(x)$ if $c(x) \geq 0$. However, as we will see in the next chapter, in practical problems it occurs frequently that the condition $c(x) \geq 0$ can not be met. Do we really need $c(x) \geq 0$? The answer is 'no'. Actually, if $c(x)$ is not 'too negative', then the operator '-△+ c(x)' can still remain 'positive' to ensure the maximum principle. These will be studied in Section 7.4, where we prove the 'Maximum Principles Based on Comparisons'.

Let ϕ be a positive function on $\bar{\Omega}$ satisfying

$$-\triangle\phi + \lambda(x)\phi \geq 0. \tag{7.4}$$

Let u be a function such that

$$\begin{cases} -\triangle u + c(x)u \geq 0 \; x \in \Omega \\ u \geq 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Theorem 7.1.5 *(Maximum Principle Based on Comparison)*

Assume that Ω is a bounded domain. If

$$c(x) > \lambda(x) \;, \forall x \in \Omega,$$

then $u \geq 0$ in Ω.

Also in Section 7.4, as consequences of Theorem 7.1.5, we derive the 'Narrow Region Principle' and the 'Decay at Infinity Principle'. These principles can be applied very conveniently in the 'Method of Moving Planes' to establish the symmetry of solutions for semi-linear elliptic equations, as we will see in later sections.

In Section 7.5, we establish a maximum principle for integral inequalities.

7.2 Weak Maximum Principles

In this section, we prove the weak maximum principles.

Theorem 7.2.1 *(Weak Maximum Principle for $-\triangle$.)*

i) If

$$-\triangle u(x) \geq 0, \quad x \in \Omega, \tag{7.6}$$

then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u. \tag{7.7}$$

ii) If

$$-\triangle u(x) \leq 0, \quad x \in \Omega, \tag{7.8}$$

then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u. \tag{7.9}$$

Proof. Here we only present the proof of part i). The entirely similar proof also works for part ii).

To better illustrate the idea, we will deal with one dimensional case and higher dimensional case separately.

First, let Ω be the interval (a,b). Then condition (7.6) becomes $u''(x) \leq 0$. This implies that the graph of $u(x)$ on (a,b) is concave downward, and therefore one can roughly see that the values of $u(x)$ in (a,b) are large or equal to the minimum value of u at the end points (See Figure 2).

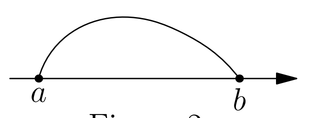


Figure 2

To prove the above observation rigorously, we first carry it out under the stronger assumption that

$$-u''(x) > 0. \tag{7.10}$$

Let $m = \min_{\partial\Omega} u$. Suppose in contrary to (7.7), there is a minimum $x^o \in (a,b)$ of u, such that $u(x^o) < m$. Then by the second order Taylor expansion of u around x^o, we must have $-u''(x^o) \leq 0$. This contradicts with the assumption (7.10).

Now for u only satisfying the weaker condition (7.6), we consider a perturbation of u:

$$u_\epsilon(x) = u(x) - \epsilon x^2.$$

Obviously, for each $\epsilon > 0$, $u_\epsilon(x)$ satisfies the stronger condition (7.10), and hence

$$\min_{\bar{\Omega}} u_\epsilon \geq \min_{\partial \Omega} u_\epsilon.$$

Now letting $\epsilon \to 0$, we arrive at (7.7).

To prove the theorem in dimensions higher than one, we need the following

Lemma 7.2.1 *(Mean Value Inequality) Let x^o be a point in Ω. Let $B_r(x^o) \subset \Omega$ be the ball of radius r center at x^o, and $\partial B_r(x^o)$ be its boundary.*

i) If $-\triangle u(x) > (=)\ 0$ for $x \in B_{r_o}(x^o)$ with some $r_o > 0$, then for any $r_o > r > 0$,

$$u(x^o) > (=) \frac{1}{|\partial B_r(x^o)|} \int_{\partial B_r(x^o)} u(x) dS. \tag{7.11}$$

It follows that, if x^o is a minimum of u in Ω, then

$$-\triangle u(x^o) \leq 0. \tag{7.12}$$

ii) If $-\triangle u(x) < 0$ for $x \in B_{r_o}(x^o)$ with some $r_o > 0$, then for any $r_o > r > 0$,

$$u(x^o) < \frac{1}{|\partial B_r(x^o)|} \int_{\partial B_r(x^o)} u(x) dS. \tag{7.13}$$

It follows that, if x^o is a maximum of u in Ω, then

$$-\triangle u(x^o) \geq 0. \tag{7.14}$$

We postpone the proof of the Lemma for a moment. This Lemma tell us that, if $-\triangle u(x) > 0$, then the value of u at the center of the small ball $B_r(x^o)$ is larger than its average value on the boundary $\partial B_r(x^o)$. Roughly speaking, the graph of u is locally somewhat concave downward. Now based on this Lemma, to prove the theorem, we first consider $u_\epsilon(x) = u(x) - \epsilon |x|^2$. Obviously,

$$-\triangle u_\epsilon = -\triangle u + 2\epsilon n > 0. \tag{7.15}$$

Hence we must have

$$\min_{\Omega} u_\epsilon \geq \min_{\partial \Omega} u_\epsilon \tag{7.16}$$

Otherwise, if there exists a minimum x^o of u in Ω, then by Lemma 7.2.1, we have $-\triangle u_\epsilon(x^o) \leq 0$. This contradicts with (7.15). Now in (7.16), letting $\epsilon \to 0$, we arrive at the desired conclusion (7.7).

This completes the proof of the Theorem.

The Proof of Lemma 7.2.1. By the Divergence Theorem,

$$\int_{B_r(x^o)} \triangle u(x) dx = \int_{\partial B_r(x^o)} \frac{\partial u}{\partial \nu} dS = r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(x^o + r\omega) dS_\omega, \tag{7.17}$$

where dS_ω is the area element of the $n-1$ dimensional unit sphere $S^{n-1} = \{\omega \mid |\omega| = 1\}$.

If $\triangle u < 0$, then by (7.17),

$$\frac{\partial}{\partial r}\{\int_{S^{n-1}} u(x^o + r\omega) dS_\omega\} < 0. \tag{7.18}$$

Integrating both sides of (7.18) from 0 to r yields

$$\int_{S^{n-1}} u(x^o + r\omega) dS_\omega - u(x^o)|S^{n-1}| < 0,$$

where $|S^{n-1}|$ is the area of S^{n-1}. It follows that

$$u(x^o) > \frac{1}{r^{n-1}|S^{n-1}|} \int_{\partial B_r(x^o)} u(x) dS.$$

This verifies (7.11).

To see (7.12), we suppose in contrary that $-\triangle u(x^o) > 0$. Then by the continuity of $\triangle u$, there exists a $\delta > 0$, such that

$$-\triangle u(x) > 0, \ \ \forall x \in B_\delta(x^o).$$

Consequently, (7.11) holds for any $0 < r < \delta$. This contradicts with the assumption that x^o is a minimum of u.

This completes the proof of the Lemma.

From the proof of Theorem 7.2.1, one can see that if we replace $-\triangle$ operator by $-\triangle + c(x)$ with $c(x) \geq 0$, then the conclusion of Theorem 7.2.1 is still true (with slight modifications). Furthermore, we can replace the Laplace operator $-\triangle$ with general uniformly elliptic operators. Let

$$D_i = \frac{\partial}{\partial x_i}, \ \ D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}.$$

Define

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x). \tag{7.19}$$

Here we always assume that $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are bounded continuous functions in $\bar{\Omega}$. We say that L is uniformly elliptic if

$$a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \ \text{ for any } x \in \Omega, \text{ any } \xi \in R^n \text{ and for some } \delta > 0.$$

Theorem 7.2.2 *(Weak Maximum Principle for L with $c(x) \equiv 0$). Let L be the uniformly elliptic operator defined above. Assume that $c(x) \equiv 0$.*

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u. \tag{7.20}$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u.$$

For $c(x) \geq 0$, the principle still applies with slight modifications.

Theorem 7.2.3 *(Weak Maximum Principle for L with $c(x) \geq 0$). Let L be the uniformly elliptic operator defined above. Assume that $c(x) \geq 0$. Let*

$$u^-(x) = \min\{u(x), 0\} \quad \text{and} \quad u^+(x) = \max\{u(x), 0\}.$$

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u^-.$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+.$$

Interested readers may find its proof in many standard books, say in [Ev], page 327.

7.3 The Hopf Lemma and Strong Maximum Principles

In the previous section, we prove a weak form of maximum principle. In the case $Lu \geq 0$, it concludes that the minimum of u attains at some point on the boundary $\partial\Omega$. However it does not exclude the possibility that the minimum may also attain at some point in the interior of Ω. In this section, we will show that this can not actually happen, that is, the minimum value of u can only be achieved on the boundary unless u is constant. This is called the "Strong Maximum Principle". We will prove it by using the following

Lemma 7.3.1 *(**Hopf Lemma**). Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Let*

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x)$$

be uniformly elliptic in Ω with $c(x) \equiv 0$. Assume that

$$Lu \geq 0 \quad \text{in } \Omega. \tag{7.21}$$

Suppose there is a ball B contained in Ω with a point $x^o \in \partial\Omega \cap \partial B$ and suppose

$$u(x) > u(x^o), \quad \forall x \in B. \tag{7.22}$$

Then for any outward directional derivative at x^o,

$$\frac{\partial u(x^o)}{\partial \nu} < 0. \tag{7.23}$$

In the case $c(x) \geq 0$, if we require additionally that $u(x^o) \leq 0$, then the same conclusion of the Hopf Lemma holds.

Proof. Without loss of generality, we may assume that B is centered at the origin with radius r. Define

$$w(x) = e^{-\alpha r^2} - e^{-\alpha |x|^2}.$$

Consider $v(x) = u(x) + \epsilon w(x)$ on the set $D = B_{\frac{r}{2}}(x^o) \cap B$ (See Figure 3).

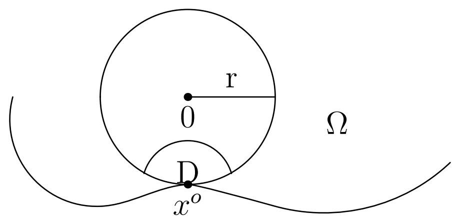


Figure 3

We will choose α and ϵ appropriately so that we can apply the Weak Maximum Principle to $v(x)$ and arrive at

$$v(x) \geq v(x^o) \;\; \forall x \in D. \tag{7.24}$$

We postpone the proof of (7.24) for a moment. Now from (7.24), we have

$$\frac{\partial v}{\partial \nu}(x^o) \leq 0, \tag{7.25}$$

Noticing that

$$\frac{\partial w}{\partial \nu}(x^o) > 0$$

We arrive at the desired inequality

$$\frac{\partial u}{\partial \nu}(x^o) < 0.$$

Now to complete the proof of the Lemma, what left to verify is (7.24). We will carry this out in two steps. First we show that

$$Lv \geq 0. \tag{7.26}$$

Hence we can apply the Weak Maximum Principle to conclude that

$$\min_{\bar{D}} v \geq \min_{\partial D} . \tag{7.27}$$

Then we show that the minimum of v on the boundary ∂D is actually attained at x^o:

$$v(x) \geq v(x^o) \;\; \forall x \in \partial D. \tag{7.28}$$

Obviously, (7.27) and (7.28) imply (7.24).

To see (7.26), we directly calculate

$$\begin{aligned} Lw &= e^{-\alpha|x|^2}\{4\alpha^2 \sum_{i,j=1}^n a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^n [a_{ii}(x) - b_i(x)x_i] - c(x)\} + c(x)e^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\{4\alpha^2 \sum_{i,j=1}^n a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^n [a_{ii}(x) - b_i(x)x_i] - c(x)\} \end{aligned} \tag{7.29}$$

By the ellipticity assumption, we have

$$\sum_{i,j=1}^n a_{ij}(x)x_i x_j \geq \delta|x|^2 \geq \delta(\frac{r}{2})^2 > 0 \quad \text{in } D. \tag{7.30}$$

Hence we can choose α sufficiently large, such that $Lw \geq 0$. This, together with the assumption $Lu \geq 0$ implies $Lv \geq 0$, and (7.27) follows from the Weak Maximum Principle.

To verify (7.28), we consider two parts of the boundary ∂D separately.

(i) On $\partial D \cap B$, since $u(x) > u(x^o)$, there exists a $c_o > 0$, such that $u(x) \geq u(x^o) + c_o$. Take ϵ small enough such that $\epsilon|w| \leq \delta$ on $\partial D \cap B$. Hence

$$v(x) \geq u(x^o) = v(x^o) \;\; \forall x \in \partial D \cap B.$$

(ii) On $\partial D \cap \partial B$, $w(x) = 0$, and by the assumption $u(x) \geq u(x^o)$, we have $v(x) \geq v(x^o)$.

This completes the proof of the Lemma.

Now we are ready to prove

Theorem 7.3.1 *(Strong Maximum Principle for L with $c(x) \equiv 0$.) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \equiv 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u attains its minimum only on $\partial\Omega$ unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u attains its maximum only on $\partial\Omega$ unless u is constant.

Proof. We prove part i) here. The proof of part ii) is similar. Let m be the minimum value of u in Ω. Set $\Sigma = \{x \in \Omega \mid u(x) = m\}$. It is relatively closed in Ω. We show that either Σ is empty or $\Sigma = \Omega$.

We argue by contradiction. Suppose Σ is a nonempty proper subset of Ω. Then we can find an open ball $B \subset \Omega \setminus \Sigma$ with a point on its boundary belonging to Σ. Actually, we can first find a point $p \in \Omega \setminus \Sigma$ such that $d(p, \Sigma) < d(p, \partial\Omega)$, then increase the radius of a small ball center at p until it hits Σ (before hitting $\partial\Omega$). Let x^o be the point at $\partial B \cap \Sigma$. Obviously we have in B

$$Lu \geq 0 \quad \text{and } u(x) > u(x^o).$$

Now we can apply the Hopf Lemma to conclude that the normal outward derivative

$$\frac{\partial u}{\partial \nu}(x^o) < 0. \tag{7.31}$$

On the other hand, x^o is an interior minimum of u in Ω, and we must have $Du(x^o) = 0$. This contradicts with (7.31) and hence completes the proof of the Theorem.

In the case when $c(x) \geq 0$, the strong principle still applies with slight modifications.

Theorem 7.3.2 *(Strong Maximum Principle for L with $c(x) \geq 0$.) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \geq 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u can not attain its non-positive minimum in the interior of Ω unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u can not attain its non-negative maximum in the interior of Ω unless u is constant.

Remark 7.3.1 *In order that the maximum principle to hold, we assume that the domain Ω be bounded. This is essential, since it guarantees the existence of maximum and minimum of u in $\bar{\Omega}$. A simple counter example is when Ω is the half space $\{x \in R^n \mid x_n > 0\}$, and $u(x, y) = x_n$. Obviously, $\triangle u = 0$, but u does not obey the maximum principle:*

$$\max_{\Omega} u \leq \max_{\partial\Omega} u.$$

Equally important is the non-negativeness of the coefficient $c(x)$. For example, set $\Omega = \{(x, y) \in R^2 \mid -\frac{\pi}{2} < x < \frac{\pi}{2}, \ -\frac{\pi}{2} < y < \frac{\pi}{2}\}$. Then $u = \cos x \cos y$ satisfies

$$\begin{cases} -\triangle u - 2u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

But, obviously, there are some points in Ω at which $u < 0$.

However, if we impose some sign restriction on u, say $u \geq 0$, then both conditions can be relaxed. A simple version of such result will be present in the next theorem.

Also, as one will see in the next section, $c(x)$ is actually allowed to be negative, but not 'too negative'.

Theorem 7.3.3 *(Maximum Principle and Hopf Lemma for not necessarily bounded domain and not necessarily non-negative $c(x)$.)*

Let Ω be a domain in R^n with smooth boundary. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and satisfies

$$\begin{cases} -\triangle u + \sum_{i=1}^n b_i(x) D_i u + c(x) u \geq 0, \;\; u(x) \geq 0, \; x \in \Omega \\ u(x) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \partial\Omega \end{cases} \tag{7.32}$$

with bounded functions $b_i(x)$ and $c(x)$. Then

i) if u vanishes at some point in Ω, then $u \equiv 0$ in Ω; and

ii) if $u \not\equiv 0$ in Ω, then on $\partial\Omega$, the exterior normal derivative $\dfrac{\partial u}{\partial \nu} < 0$.

To prove the Theorem, we need the following Lemma concerning eigenvalues.

Lemma 7.3.2 *Let λ_1 be the first positive eigenvalue of*

$$\begin{cases} -\triangle \phi = \lambda_1 \phi(x) \;\; x \in B_1(0) \\ \phi(x) = 0 \qquad\quad\;\; x \in \partial B_1(0) \end{cases} \tag{7.33}$$

with the corresponding eigenfunction $\phi(x) > 0$. Then for any $\rho > 0$, the first positive eigenvalue of the problem on $B_\rho(0)$ is $\dfrac{\lambda_1}{\rho^2}$. More precisely, if we let $\psi(x) = \phi(\frac{x}{\rho})$, then

$$\begin{cases} -\triangle \psi = \dfrac{\lambda_1}{\rho^2} \psi(x) \;\; x \in B_\rho(0) \\ \psi(x) = 0 \qquad\qquad\;\; x \in \partial B_\rho(0) \end{cases} \tag{7.34}$$

The proof is straight forward, and will be left for the readers.

The Proof of Theorem 7.3.3.

i) Suppose that $u = 0$ at some point in Ω, but $u \not\equiv 0$ on Ω. Let

$$\Omega_+ = \{x \in \Omega \mid u(x) > 0\}.$$

Then by the regularity assumption on u, Ω_+ is an open set with C^2 boundary. Obviously,

$$u(x) = 0 \; , \;\; \forall \, x \in \partial\Omega_+.$$

Let x^o be a point on $\partial\Omega_+$, but not on $\partial\Omega$. Then for $\rho > 0$ sufficiently small, one can choose a ball $B_{\rho/2}(\bar{x}) \subset \Omega_+$ with x^o as its boundary point. Let ψ be the positive eigenfunction of the eigenvalue problem (7.34) on $B_\rho(x^o)$ corresponding to the eigenvalue $\frac{\lambda_1}{\rho^2}$. Obviously, $B_\rho(x^o)$ completely covers $B_{\rho/2}(\bar{x})$. Let $v = \frac{u}{\psi}$. Then from (7.32), it is easy to deduce that

$$
\begin{aligned}
0 &\leq -\triangle v - 2\bigtriangledown v \cdot \frac{\bigtriangledown \psi}{\psi} + \sum_{i=1}^{n} b_i(x) D_i v + \left(\frac{-\triangle \psi}{\psi} + \sum_{i=1}^{n} \frac{D_i \psi}{\psi} + c(x) \right) v \\
&\equiv -\triangle v + \sum_{i=1}^{n} \tilde{b}_i(x) D_i v + \tilde{c}(x) v.
\end{aligned}
$$

Let ϕ be the positive eigenfunction of the eigenvalue problem (7.33) on B_1, then

$$
\tilde{c}(x) = \frac{\lambda_1}{\rho^2} + \frac{1}{\rho} \sum_{i=1}^{n} \frac{D_i \phi}{\phi} + c(x).
$$

This allows us to choose ρ sufficiently small so that $\tilde{c}(x) \geq 0$. Now we can apply *Hopf Lemma* to conclude that, the outward normal derivative at the boundary point x^o of $B_{\rho/2}(\bar{x})$,

$$
\frac{\partial v}{\partial \nu}(x^o) < 0, \tag{7.35}
$$

because

$$
v(x) > 0 \ \forall x \in B_{\rho/2}(\bar{x}) \ \text{ and } v(x^o) = \frac{u(x^o)}{\psi(x^o)} = 0.
$$

On the other hand, since x^o is also a minimum of v in the interior of Ω, we must have

$$
\bigtriangledown v(x^o) = 0.
$$

This contradicts with (7.35) and hence proves part i) of the Theorem.

ii) The proof goes almost the same as in part i) except we consider the point x^o on $\partial\Omega$ and the ball $B_{\rho/2}(\bar{x})$ is in Ω with $x^o \in \partial B_{\rho/2}(\bar{x})$. Then for the outward normal derivative of u, we have

$$
\frac{\partial u}{\partial \nu}(x^o) = \frac{\partial v}{\partial \nu}(x^o)\psi(x^o) + v(x^o)\frac{\partial \psi}{\partial \nu}(x^o) = \frac{\partial v}{\partial \nu}(x^o)\psi(x^o) < 0.
$$

Here we have used a well-known fact that the eigenfunction ψ on $B_\rho(x^o)$ is radially symmetric about the center x^o, and hence $\bigtriangledown \psi(x^o) = 0$. This completes the proof of the Theorem.

7.4 Maximum Principles Based on Comparisons

In the previous section, we show that if $(-\triangle + c(x))u \geq 0$, then the maximum principle, i.e. (7.20), applies. There, we required $c(x) \geq 0$. We can think $-\triangle$

as a 'positive' operator, and the maximum principle holds for any 'positive' operators. For $c(x) \geq 0$, $-\triangle + c(x)$ is also 'positive'. Do we really need $c(x) \geq 0$ here? To answer the question, let us consider the Dirichlet eigenvalue problem of $-\triangle$:

$$\begin{cases} -\triangle\phi - \lambda\phi(x) = 0 & x \in \Omega \\ \phi(x) = 0 & x \in \partial\Omega. \end{cases} \tag{7.36}$$

We notice that the eigenfunction ϕ corresponding to the first positive eigenvalue λ_1 is either positive or negative in Ω. That is, the solutions of (7.36) with $\lambda = \lambda_1$ obey Maximum Principle, that is, the maxima or minima of ϕ are attained only on the boundary $\partial\Omega$. This suggests that, to ensure the Maximum Principle, $c(x)$ need not be nonnegative, it is allowed to be as negative as $-\lambda_1$. More precisely, we can establish the following more general maximum principle based on comparison.

Theorem 7.4.1 *Assume that Ω is a bounded domain. Let ϕ be a positive function on $\bar{\Omega}$ satisfying*

$$-\triangle\phi + \lambda(x)\phi \geq 0. \tag{7.37}$$

Assume that u is a solution of

$$\begin{cases} -\triangle u + c(x)u \geq 0 & x \in \Omega \\ u \geq 0 & \text{on } \partial\Omega. \end{cases} \tag{7.38}$$

If

$$c(x) > \lambda(x)\ , \quad \forall x \in \Omega, \tag{7.39}$$

then $u \geq 0$ in Ω.

Proof. We argue by contradiction. Suppose that $u(x) < 0$ somewhere in Ω. Let $v(x) = \dfrac{u(x)}{\phi(x)}$. Then since $\phi(x) > 0$, we must have $v(x) < 0$ somewhere in Ω. Let $x^o \in \Omega$ be a minimum of $v(x)$. By a direct calculation, it is easy to verify that

$$-\triangle v = 2\bigtriangledown v \cdot \frac{\bigtriangledown\phi}{\phi} + \frac{1}{\phi}(-\triangle u + \frac{\triangle\phi}{\phi}u). \tag{7.40}$$

On one hand, since x^o is a minimum, we have

$$-\triangle v(x^o) \leq 0 \quad \text{and } \bigtriangledown v(x^o) = 0. \tag{7.41}$$

While on the other hand, by (7.37), (7.38), and (7.39), and taking into account that $u(x^o) < 0$, we have, at point x^o,

$$\begin{aligned} -\triangle u + \frac{\triangle\phi}{\phi}u(x^o) &\geq -\triangle u + \lambda(x^o)u(x^o) \\ &> -\triangle u + c(x^o)u(x^o) \geq 0. \end{aligned}$$

This is an obvious contradiction with (7.40) and (7.41), and thus completes the proof of the Theorem.

Remark 7.4.1 *From the proof, one can see that conditions (7.37) and (7.39) are required only at the points where v attains its minimum, or at points where u is negative.*

The Theorem is also valid on an unbounded domains if u is “nonnegative” at infinity:

Theorem 7.4.2 *If Ω is an unbounded domain, besides condition (7.38), we assume further that*

$$\liminf_{|x| \longrightarrow \infty} u(x) \geq 0. \tag{7.42}$$

Then $u \geq 0$ in Ω.

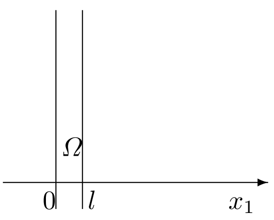
Proof. Still consider the same $v(x)$ as in the proof of Theorem 7.4.1. Now condition (7.42) guarantees that the minima of $v(x)$ do not “leak” away to infinity. Then the rest of the arguments are exactly the same as in the proof of Theorem 7.4.1.

For convenience in applications, we provide two typical situations where there exist such functions ϕ and $c(x)$ satisfying condition (7.37) and (7.39), so that the Maximum Principle Based on Comparison applies:
i) Narrow regions, and
ii) $c(x)$ decays fast enough at ∞.

i) Narrow Regions. When

$$\Omega = \{x \mid 0 < x_1 < l\}$$

is a narrow region with width l as shown:



We can choose $\phi(x) = \sin(\frac{x_1 + \epsilon}{l})$. Then it is easy to see that $-\triangle\phi = (\frac{1}{l})^2\phi$, where $\lambda(x) = \dfrac{-1}{l^2}$ can be very negative when l is sufficiently small.

Corollary 7.4.1 *(*Narrow Region Principle.*) If u satisfies (7.38) with bounded function $c(x)$. Then when the width l of the region Ω is sufficiently small, $c(x)$ satisfies (7.39), i.e. $c(x) > \lambda(x) = \dfrac{-1}{l^2}$. Hence we can directly apply Theorem 7.4.1 to conclude that $u \geq 0$ in Ω, provided* $\liminf_{|x| \rightarrow \infty} u(x) \geq 0$.

ii) Decay at Infinity. In dimension $n \geq 3$, one can choose some positive number $q < n - 2$, and let $\phi(x) = \frac{1}{|x|^q}$ Then it is easy to verify that

$$-\triangle\phi = \frac{q(n-2-q)}{|x|^2}\phi.$$

In the case $c(x)$ decays fast enough near infinity, we can adapt the proof of Theorem 7.4.1 to derive

Corollary 7.4.2 (Decay at Infinity) *Assume there exist $R > 0$, such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2}, \quad \forall |x| > R. \tag{7.43}$$

Suppose

$$\lim_{|x| \longrightarrow \infty} \frac{u(x)}{|x|^q} = 0.$$

Let Ω be a region containing in $B_R^C(0) \equiv R^n \setminus B_R(0)$. If u satisfies (7.38) on $\bar{\Omega}$, then

$$u(x) \geq 0 \quad \textit{for all } x \in \Omega.$$

Remark 7.4.2 *From Remark 7.4.1, one can see that actually condition (7.43) is only required at points where u is negative.*

Remark 7.4.3 *Although Theorem 7.4.1 as well as its Corollaries are stated in linear forms, they can be easily applied to a nonlinear equation, for example,*

$$-\triangle u - |u|^{p-1}u = 0 \quad x \in R^n. \tag{7.44}$$

Assume that the solution u decays near infinity at the rate of $\frac{1}{|x|^s}$ with $s(p-1) > 2$. Let $c(x) = -|u(x)|^{p-1}$. Then for R sufficiently large, and for the region Ω as stated in Corollary 7.4.2, $c(x)$ satisfies (7.43) in Ω. If further assume that

$$u \mid_{\partial\Omega} \geq 0,$$

then we can derive from Corollary 7.4.2 that $u \geq 0$ in the entire region Ω.

7.5 A Maximum Principle for Integral Equations

In this section, we introduce a maximum principle for integral equations.

Let Ω be a region in R^n, may or may not be bounded. Assume

$$K(x,y) \geq 0, \quad \forall (x,y) \in \Omega \times \Omega.$$

Define the integral operator T by

$$(Tf)(x) = \int_\Omega K(x,y)f(y)dy.$$

Let $\|\cdot\|$ be a norm in a Banach space satisfying

$$\|f\| \leq \|g\|, \quad \text{whenever } 0 \leq f(x) \leq g(x) \text{ in } \Omega. \tag{7.45}$$

There are many norms that satisfy (7.45), such as $L^p(\Omega)$ norms, $L^\infty(\Omega)$ norm, and $C(\Omega)$ norm.

Theorem 7.5.1 *Suppose that*

$$\|Tg\| \leq \theta \|g\| \quad \textit{with some } 0 < \theta < 1 \;\; \forall g. \tag{7.46}$$

If

$$f(x) \leq (Tf)(x), \tag{7.47}$$

then

$$f(x) \leq 0 \;\; \textit{for all } x \in \Omega. \tag{7.48}$$

Proof. Define

$$\Omega_+ = \{x \in \Omega \mid f(x) > 0\},$$

and

$$f^+(x) = \max\{0, f(x)\}, \;\; f^-(x) = \min\{0, f(x)\}.$$

Then it is easy to see that for any $x \in \Omega_+$,

$$0 \leq f^+(x) \leq (Tf)(x) = (Tf)^+(x) + (Tf)^-(x) \leq (Tf)^+(x) \tag{7.49}$$

While in the rest of the region Ω, we have $f^+(x) = 0$ and $(Tf)^+(x) \geq 0$ by the definitions. Therefore, (7.49) holds for any $x \in \Omega$. It follows from (7.49) that

$$\|f^+\| \leq \|(Tf)^+\| \leq \theta \|f^+\|.$$

Since $\theta < 1$, we must have

$$\|f^+\| = 0,$$

which implies that $f^+(x) \equiv 0$, and hence $f(x) \leq 0$. This completes the proof of the Theorem.

Recall that in Section 5, after introducing the "*Maximum Principle Based on Comparison*" for Partial Differential Equations, we explained how it can be applied to the situations of "*Narrow Regions*" and "*Decay at Infinity*". Parallel to this, we have similar applications for integral equations. Let α be a number between 0 and n. Define

$$(Tf)(x) = \int_\Omega \frac{1}{|x-y|^{n-\alpha}} c(y) f(y) dy.$$

Assume that f satisfies the integral equation

$$f = Tf \quad \text{in } \Omega,$$

or more generally, the integral inequality

$$f \leq Tf \quad \text{in } \Omega. \tag{7.50}$$

Further assume that

$$\|Tf\|_{L^p(\Omega)} \leq C\|c(y)\|_{L^\tau(\Omega)}\|f\|_{L^p(\Omega)}, \tag{7.51}$$

for some $p, \tau > 1$.

If we have some right integrability condition on $c(y)$, then we can derived, from Theorem 7.5.1, a maximum principle that will be applied to "*Narrow Regions*" and "*Near Infinity*". More precisely, we have

Corollary 7.5.1 *Assume that $c(y) \geq 0$ and $c(y) \in L^\tau(R^n)$. Let $f \in L^p(R^n)$ be a nonnegative function satisfying (7.50) and (7.51). Then there exist positive numbers R_o and ϵ_o depending on $c(y)$ only, such that*

$$\textit{if } \mu(\Omega \cap B_{R_o}(0)) \leq \epsilon_o, \quad \textit{then } f^+ \equiv 0 \ \ \textit{in } \Omega.$$

where $\mu(D)$ is the measure of the set D.

Proof. Since $c(y) \in L^\tau(R^n)$, by Lebesgue integral theory, when the measure of the intersection of Ω with $B_{R_o}(0)$ is sufficiently small, we can make the integral $\int_\Omega |c(y)|^\tau dy$ as small as we wish, and thus to obtain

$$C\|c(y)\|_{L^\tau(\Omega)} < 1.$$

Now it follows from Theorem 7.5.1 that

$$f^+(x) \equiv 0 \,, \forall\, x \in \Omega.$$

This completes the proof of the Corollary.

Remark 7.5.1 *One can see that the condition $\mu(\Omega \cap B_{R_o}(0)) \leq \epsilon_o$ in the Corollary is satisfied in the following two situations.*

i) Narrow Regions*: The width of Ω is very small.*

ii) Near Infinity*: Say, $\Omega = B_R^C(0)$, the complement of the ball $B_R(0)$, with sufficiently large R.*

As an immediate application of this "*Maximum Principle*", we study an integral equation in the next section. We will use the method of moving planes to obtain the radial symmetry and monotonicity of the positive solutions.

8

Methods of Moving Planes and of Moving Spheres

The Method of Moving Planes (MMP) was invented by the Soviet mathematician Alexanderoff in the early 1950s. Decades later, it was further developed by Serrin [Se], Gidas, Ni, and Nirenberg [GNN], Caffarelli, Gidas, and Spruck [CGS], Li [Li], Chen and Li [CL] [CL1], Chang and Yang [CY], and many others. This method has been applied to free boundary problems, semi-linear partial differential equations, and other problems. Particularly for semi-linear partial differential equations, there have seen many significant contributions. We refer to the paper of Frenkel [F] for more descriptions on the method.

The Method of Moving Planes and its variant–the Method of Moving Spheres–have become powerful tools in establishing symmetries and monotonicity for solutions of partial differential equations. They can also be used to obtain a priori estimates, to derive useful inequalities, and to prove nonexistence of solutions.

From the previous chapter, we have seen the beauty and power of maximum principles. The MMP greatly enhances the power of maximum principles. Roughly speaking, the MMP is a continuous way of repeated applications of maximum principles. During this process, the maximum principle has been used infinitely many times; and the advantage is that each time we only need to use the maximum principle in a very narrow region. From the previous chapter, one can see that, in such a narrow region, even if the coefficients of the equation are not 'good,' the maximum principle can still be applied. In the authors' research practice, we also introduced a form of maximum principle at infinity to the MMP and therefore simplified many proofs and extended the results in more natural ways. We recommend the readers study this part carefully, so that they will be able to apply it to their own research.

It is well-known that by using a Green's function, one can change a differential equation into an integral equation, and under certain conditions, they are equivalent. To investigate the symmetry and monotonicity of integral equations, the authors, together with Ou, created an integral form of MMP. Instead of using local properties (say differentiability) of a differential equation, they employed the global properties of the solutions of integral equations.

In this chapter, we will apply the Method of Moving Planes and their variant–the Method of Moving Spheres– to study semi-linear elliptic equations and integral equations. We will establish symmetry, monotonicity, a priori estimates, and non-existence of the solutions. During the process of Moving Planes, the Maximum Principles introduced in the previous chapter are applied in innovative ways.

In Section 8.2, we will establish radial symmetry and monotonicity for the solutions of the following three semi-linear elliptic problems

$$\begin{cases} -\triangle u = f(u) \;\; x \in B_1(0) \\ u = 0 \qquad\qquad \text{on } \partial B_1(0); \end{cases}$$

$$-\triangle u = u^{\frac{n+2}{n-2}}(x) \;\; x \in R^n \;\; n \geq 3;$$

and

$$-\triangle u = e^{u(x)} \;\; x \in R^2.$$

During the moving of planes, the Maximum Principles Base on Comparison will play a major role. In particular, the Narrow Region Principle and the Decay at Infinity Principle will be used repeatedly in dealing with the three examples.

In Section 8.3, we will apply the Method of Moving Planes in a 'local way' to obtain a priori estimates on the solutions of the prescribing scalar curvature equation on a compact Riemannian manifold M

$$-\frac{4(n-1)}{n-2}\triangle_o u + R_o(x)u = R(x)u^{\frac{n+2}{n-2}}, \quad \text{in } M.$$

We alow the function $R(x)$ to change signs. In this situation, the traditional blowing-up analysis fails near the set where $R(x) = 0$. We will use the Method of Moving Planes in an innovative way to obtain a priori estimates. Since the Method of Moving Planes can not be applied to the solution u directly, we introduce an auxiliary function to circumvent this difficulty.

In Section 8.4, we use the Method of Moving Spheres to prove a non-existence of solutions for the prescribing Gaussian and scalar curvature equations

$$-\triangle u + 2 = R(x)e^u,$$

and

$$-\triangle u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} R(x) u^{\frac{n+2}{n-2}}$$

on S^2 and on S^n $(n \geq 3)$, respectively. We prove that if the function $R(x)$ is rotationally symmetric and monotone in the region where it is positive, then both equations admit no solution. This provides a stronger necessary condition than the well known Kazdan-Warner condition, and it also becomes a sufficient condition for the existence of solutions in most cases.

In Section 8.5, as an application of the maximum principle for integral equations introduced in Section 7.5, we study the integral equation in R^n

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u^{\frac{n+\alpha}{n-\alpha}}(y) dy,$$

for any real number α between 0 and n. It arises as an Euler-Lagrange equation for a functional in the context of the Hardy-Littlewood-Sobolev inequalities. Due to the different nature of the integral equation, the traditional Method of Moving Planes does not work. Hence we exploit its global property and develop a new idea–the Integral Form of the Method of Moving Planes to obtain the symmetry and monotonicity of the solutions. The Maximum Principle for Integral Equations established in Chapter 7 is combined with the estimates of various integral norms to carry on the moving of planes.

8.1 Outline of the Method of Moving Planes

To outline how the Method of Moving Planes works, we take the Euclidian space R^n for an example. Let u be a positive solution of a certain partial differential equation. If we want to prove that it is symmetric and monotone in a given direction, we may assign that direction as x_1 axis. For any real number λ, let

$$T_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in R^n \mid x_1 = \lambda\}.$$

This is a plane perpendicular to x_1-axis and the plane that we will move with. Let Σ_λ denote the region to the left of the plane, i.e.

$$\Sigma_\lambda = \{x \in R^n \mid x_1 < \lambda\}.$$

Let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n),$$

the reflection of the point $x = (x_1, \cdots, x_n)$ about the plane T_λ (See Figure 1).

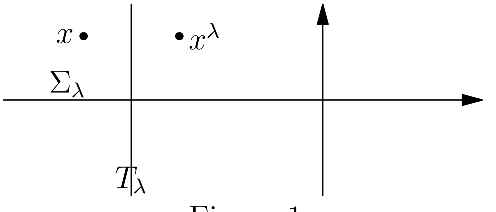


Figure 1

We compare the values of the solution u at point x and x^λ, and we want to show that u is symmetric about some plane T_{λ_o}. To this end, let

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

In order to show that, there exists some λ_o, such that

$$w_{\lambda_o}(x) \equiv 0 \ , \ \forall x \in \Sigma_{\lambda_o},$$

we generally go through the following two steps.

Step 1. We first show that for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0 \ , \ \forall x \in \Sigma_\lambda. \tag{8.1}$$

Then we are able to start off from this neighborhood of $x_1 = -\infty$, and move the plane T_λ along the x_1 direction to the right as long as the inequality (8.1) holds.

Step 2. We continuously move the plane this way up to its limiting position. More precisely, we define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda\}.$$

We prove that u is symmetric about the plane T_{λ_o}, that is $w_{\lambda_o}(x) \equiv 0$ for all $x \in \Sigma_{\lambda_o}$. This is usually carried out by a contradiction argument. We show that if $w_{\lambda_o}(x) \not\equiv 0$, then there would exist $\lambda > \lambda_o$, such that (8.1) holds, and this contradicts with the definition of λ_o.

From the above illustration, one can see that the key to the Method of Moving Planes is to establish inequality (8.1), and for partial differential equations, maximum principles are powerful tools for this task. While for integral equations, we use a different idea. We estimate a certain norm of w_λ on the set

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$$

where the inequality (8.1) is violated. We show that this norm must be zero, and hence Σ_λ^- is empty.

8.2 Applications of the Maximum Principles Based on Comparisons

In this section, we study some semi-linear elliptic equations. We will apply the method of moving planes to establish the symmetry of the solutions. The essence of the method of moving planes is the application of various maximum principles. In the proof of each theorem, the readers will see vividly how the Maximum Principles Based on Comparisons are applied to *narrow regions* and to *solutions with decay at infinity.*

8.2.1 Symmetry of Solutions in a Unit Ball

We first begin with an elegant result of Gidas, Ni, and Nirenberg [GNN1]:

Theorem 8.2.1 *Assume that $f(\cdot)$ is a Lipschitz continuous function such that*

$$|f(p) - f(q)| \leq C_o|p - q| \tag{8.2}$$

for some constant C_o. Then every positive solution u of

$$\begin{cases} -\triangle u = f(u) \;\; x \in B_1(0) \\ u = 0 \qquad\qquad \text{on } \partial B_1(0). \end{cases} \tag{8.3}$$

is radially symmetric and monotone decreasing about the origin.

Proof.

As shown on Figure 4 below, let $T_\lambda = \{x \mid x_1 = \lambda\}$ be the plane perpendicular to the x_1 axis. Let Σ_λ be the part of $B_1(0)$ which is on the left of the plane T_λ. For each $x \in \Sigma_\lambda$, let x^λ be the reflection of the point x about the plane T_λ, more precisely, $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$.

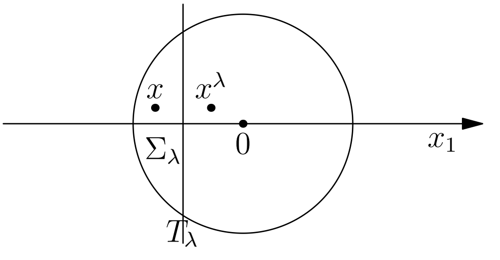


Figure 4

We compare the values of the solution u on Σ_λ with those on its reflection. Let

$$u_\lambda(x) = u(x^\lambda), \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x).$$

Then it is easy to see that u_λ satisfies the same equation as u does. Applying the Mean Value Theorem to $f(u)$, one can verify that w_λ satisfies

$$\triangle w_\lambda + C(x,\lambda) w_\lambda(x) = 0, \;\; x \in \Sigma_\lambda,$$

where

$$C(x,\lambda) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)},$$

and by condition (8.2),

$$|C(x,\lambda)| \le C_o. \tag{8.4}$$

Step 1: Start Moving the Plane.

We start from the near left end of the region. Obviously, for λ sufficiently close to -1, Σ_λ is a narrow (in x_1 direction) region, and on $\partial\Sigma_\lambda$, $w_\lambda(x) \ge 0$. (On T_λ, $w_\lambda(x) = 0$; while on the curve part of $\partial\Sigma_\lambda$, $w_\lambda(x) > 0$ since $u > 0$ in $B_1(0)$.)

Now we can apply the "*Narrow Region Principle*" (Corollary 7.4.1) to conclude that, for λ close to -1,

$$w_\lambda(x) \ge 0, \quad \forall x \in \Sigma_\lambda. \tag{8.5}$$

This provides a starting point for us to move the plane T_λ

Step 2: Move the Plane to Its Right Limit.

We now increase the value of λ continuously, that is, we move the plane T_λ to the right as long as the inequality (8.5) holds. We show that, by moving this way, the plane will not stop before hitting the origin. More precisely, let

$$\bar\lambda = \sup\{\lambda \mid w_\lambda(x) \ge 0, \; \forall x \in \Sigma_\lambda\},$$

we first claim that

$$\bar\lambda \ge 0. \tag{8.6}$$

Otherwise, we will show that the plane can be further moved to the right by a small distance, and this would contradicts with the definition of $\bar\lambda$. In fact, if $\bar\lambda < 0$, then the image of the curved surface part of $\partial\Sigma_{\bar\lambda}$ under the reflection about $T_{\bar\lambda}$ lies inside $B_1(0)$, where $u(x) > 0$ by assumption. It follows that, on this part of $\partial\Sigma_{\bar\lambda}$, $w_{\bar\lambda}(x) > 0$. By the Strong Maximum Principle, we deduce that

$$w_{\bar\lambda}(x) > 0$$

in the interior of $\Sigma_{\bar\lambda}$.

Let d_o be the maximum width of narrow regions that we can apply the "*Narrow Region Principle*". Choose a small positive number δ, such that $\delta \le \frac{d_o}{2}, -\bar\lambda$. We consider the function $w_{\bar\lambda+\delta}(x)$ on the narrow region (See Figure 5):

$$\Omega_\delta = \Sigma_{\bar\lambda+\delta} \cap \{x \mid x_1 > \bar\lambda - \frac{d_o}{2}\}.$$

It satisfies

$$\begin{cases} \triangle w_{\bar\lambda+\delta} + C(x, \bar\lambda + \delta) w_{\bar\lambda+\delta} = 0 & x \in \Omega_\delta \\ w_{\bar\lambda+\delta}(x) \geq 0 & x \in \partial\Omega_\delta. \end{cases} \tag{8.7}$$

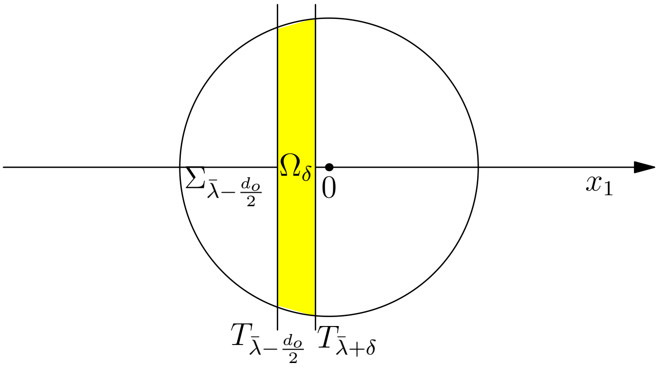


Figure 5

The equation is obvious. To see the boundary condition, we first notice that it is satisfied on the two curved parts and one flat part where $x_1 = \bar\lambda + \delta$ of the boundary $\partial\Omega_\delta$ due to the definition of $w_{\bar\lambda+\delta}$. To see that it is also true on the rest of the boundary where $x_1 = \bar\lambda - \frac{d_o}{2}$, we use continuity argument. Notice that on this part, $w_{\bar\lambda}$ is positive and bounded away from 0. More precisely and more generally, there exists a constant $c_o > 0$, such that

$$w_{\bar\lambda}(x) \geq c_o\ , \quad \forall x \in \Sigma_{\bar\lambda-\frac{d_o}{2}}.$$

Since w_λ is continuous in λ, for δ sufficiently small, we still have

$$w_{\bar\lambda+\delta}(x) \geq 0\ , \quad \forall x \in \Sigma_{\bar\lambda-\frac{d_o}{2}}.$$

Hence in particular, the boundary condition in (8.7) holds for such small δ. Now we can apply the "*Narrow Region Principle*" to conclude that

$$w_{\bar\lambda+\delta}(x) \geq 0, \quad \forall x \in \Omega_\delta.$$

And therefore,

$$w_{\bar\lambda+\delta}(x) \geq 0, \quad \forall x \in \Sigma_{\bar\lambda+\delta}.$$

This contradicts with the definition of $\bar\lambda$ and thus establishes (8.6).

(8.6) implies that

$$u(-x_1, x') \leq u(x_1, x')\ , \forall x_1 \geq 0, \tag{8.8}$$

where $x' = (x_2, \cdots, x_n)$.

We then start from λ close to 1 and move the plane T_λ toward the left. Similarly, we obtain

$$u(-x_1, x') \geq u(x_1, x') \ , \forall x_1 \geq 0, \tag{8.9}$$

Combining two opposite inequalities (8.8) and (8.9), we see that $u(x)$ is symmetric about the plane T_0. Since we can place x_1 axis in any direction, we conclude that $u(x)$ must be radially symmetric about the origin. Also the monotonicity easily follows from the argument. This completes the proof.

8.2.2 Symmetry of Solutions of $-\triangle u = u^p$ in R^n

In an elegant paper of Gidas, Ni, and Nirenberg [2], an interesting results is the symmetry of the positive solutions of the semi-linear elliptic equation:

$$\triangle u + u^p = 0 \ , \ \ x \in R^n, n \geq 3. \tag{8.10}$$

They proved

Theorem 8.2.2 *For $p = \frac{n+2}{n-2}$, all the positive solutions of (8.10) with reasonable behavior at infinity, namely*

$$u = O(\frac{1}{|x|^{n-2}}),$$

are radially symmetric and monotone decreasing about some point, and hence assume the form

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^o|^2)^{\frac{n-2}{2}}} \ \ \text{for } \lambda > 0 \text{ and for some } x^o \in R^n.$$

This uniqueness result, as was pointed out by R. Schoen, is in fact equivalent to the geometric result due to Obata [O]: A Riemannian metric on S^n which is conformal to the standard one and having the same constant scalar curvature is the pull back of the standard one under a conformal map of S^n to itself. Recently, Caffarelli, Gidas and Spruck [CGS] removed the decay assumption $u = O(|x|^{2-n})$ and proved the same result. In the case that $1 \leq p < \frac{n+2}{n-2}$, Gidas and Spruck [GS] showed that the only non-negative solution of (8.10) is identically zero. Then, in the authors paper [CL1], a simpler and more elementary proof was given for almost the same result:

Theorem 8.2.3 **i)** *For $p = \frac{n+2}{n-2}$, every positive C^2 solution of (8.10) must be radially symmetric and monotone decreasing about some point, and hence assumes the form*

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^o|^2)^{\frac{n-2}{2}}} \ \ \text{for some } \lambda > 0 \ \text{ and } x^o \in R^n.$$

ii) *For $p < \frac{n+2}{n-2}$, the only nonnegative solution of (8.10) is identically zero.*

The proof of Theorem 8.2.2 is actually included in the more general proof of the first part of Theorem 8.2.3. However, to better illustrate the idea, we will first present the proof of Theorem 8.2.2 (mostly in our own idea). And the readers will see vividly, how the "*Decay at Infinity*" principle is applied here.

Proof of Theorem 8.2.2.

Define

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) \in R^n \mid x_1 < \lambda\}, \; T_\lambda = \partial \Sigma_\lambda$$

and let x^λ be the reflection point of x about the plane T_λ, i.e.

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n).$$

(See the previous Figure 1.)

Let

$$u_\lambda(x) = u(x^\lambda), \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x).$$

The proof consists of three steps. In the first step, we start from the very left end of our region R^n, that is near $x_1 = -\infty$. We will show that, for λ sufficiently negative,

$$w_\lambda(x) \geq 0, \;\; \forall x \in \Sigma_\lambda. \tag{8.11}$$

Here, the "*Decay at Infinity*" is applied.

Then in the second step, we will move our plane T_λ in the the x_1 direction toward the right as long as inequality (8.11) holds. The plane will stop at some limiting position, say at $\lambda = \lambda_o$. We will show that

$$w_{\lambda_o}(x) \equiv 0, \;\; \forall x \in \Sigma_{\lambda_o}.$$

This implies that the solution u is symmetric and monotone decreasing about the plane T_{λ_o}. Since x_1-axis can be chosen in any direction, we conclude that u must be radially symmetric and monotone about some point.

Finally, in the third step, using the uniqueness theory in Ordinary Differential Equations, we will show that the solutions can only assume the given form.

Step 1. Prepare to Move the Plane from Near $-\infty$.

To verify (8.11) for λ sufficiently negative, we apply the maximum principle to $w_\lambda(x)$. Write $\tau = \frac{n+2}{n-2}$. By the definition of u_λ, it is easy to see that, u_λ satisfies the same equation as u does. Then by the *Mean Value Theorem*, it is easy to verify that

$$-\triangle w_\lambda = u_\lambda^\tau(x) - u^\tau(x) = \tau \psi_\lambda^{\tau-1}(x) w_\lambda(x). \tag{8.12}$$

where $\psi_\lambda(x)$ is some number between $u_\lambda(x)$ and $u(x)$. Recalling the "*Maximum Principle Based on Comparison*" (Theorem 7.4.1), we see here $c(x) =$

$-\tau\psi_\lambda^{\tau-1}(x)$. By the "*Decay at Infinity*" argument (Corollary 7.4.2), it suffice to check the decay rate of $\psi_\lambda^{\tau-1}(x)$, and more precisely, only at the points $\tilde{x}$ where w_λ is negative (see Remark 7.4.2). Apparently at these points,

$$u_\lambda(\tilde{x}) < u(\tilde{x}),$$

and hence

$$0 \leq u_\lambda(\tilde{x}) \leq \psi_\lambda(\tilde{x}) \leq u(\tilde{x}).$$

By the decay assumption of the solution

$$u(x) = O\left(\frac{1}{|x|^{n-2}}\right),$$

we derive immediately that

$$\psi_\lambda^{\tau-1}(\tilde{x}) = O\left(\left(\frac{1}{|\tilde{x}|}\right)^{\frac{4}{n-2}}\right) = O\left(\frac{1}{|\tilde{x}|^4}\right).$$

Here the power of $\frac{1}{|\tilde{x}|}$ is greater than two, which is what (actually more than) we desire for. Therefore, we can apply the "*Maximum Principle Based on Comparison*" to conclude that for λ sufficiently negative ($|\tilde{x}|$ sufficiently large), we must have (8.11). This completes the preparation for the moving of planes.

Step 2. Move the Plane to the Limiting Position to Derive Symmetry.

Now we can move the plane T_λ toward right, i.e., increase the value of λ, as long as the inequality (8.11) holds. Define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\}.$$

Obviously, $\lambda_o < +\infty$, due to the asymptotic behavior of u near $x_1 = +\infty$. We claim that

$$w_{\lambda_o}(x) \equiv 0, \ \ \forall x \in \Sigma_{\lambda_o}. \tag{8.13}$$

Otherwise, by the "*Strong Maximum Principle*" on unbounded domains (see Theorem 7.3.3), we have

$$w_{\lambda_o}(x) > 0 \ \text{ in the interior of } \Sigma_{\lambda_o}. \tag{8.14}$$

We show that the plane T_{λ_o} can still be moved a small distance to the right. More precisely, there exists a $\delta_o > 0$ such that, for all $0 < \delta < \delta_o$, we have

$$w_{\lambda_o+\delta}(x) \geq 0 \ , \ \ \forall x \in \Sigma_{\lambda_o+\delta}. \tag{8.15}$$

This would contradict with the definition of λ_o, and hence (8.13) must hold.

Recall that in the last section, we use the *"Narrow Region Principle "* to derive (8.15). Unfortunately, it can not be applied in this situation, because the "narrow region" here is unbounded, and we are not able to guarantee that w_{λ_o} is bounded away from 0 on the left boundary of the "narrow region".

To overcome this difficulty, we introduce a new function

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)},$$

where

$$\phi(x) = \frac{1}{|x|^q} \text{ with } 0 < q < n-2.$$

Then it is a straight forward calculation to verify that

$$-\triangle \bar{w}_\lambda = 2\bigtriangledown \bar{w}_\lambda \cdot \frac{\bigtriangledown \phi}{\phi} + \left(-\triangle w_\lambda + \frac{\triangle \phi}{\phi} w_\lambda\right) \frac{1}{\phi} \tag{8.16}$$

We have

Lemma 8.2.1 *There exists a $R_o > 0$ (independent of λ), such that if x^o is a minimum point of $\bar{w}_\lambda$ and $\bar{w}_\lambda(x^o) < 0$, then $|x^o| < R_o$.*

We postpone the proof of the Lemma for a moment. Now suppose that (8.15) is violated for any $\delta > 0$. Then there exists a sequence of numbers $\{\delta_i\}$ tending to 0 and for each i, the corresponding negative minimum x^i of $w_{\lambda_o+\delta_i}$. By Lemma 8.2.1, we have

$$|x^i| \leq R_o \, , \quad \forall \, i = 1, 2, \cdots .$$

Then, there is a subsequence of $\{x^i\}$ (still denoted by $\{x^i\}$) which converges to some point $x^o \in R^n$. Consequently,

$$\bigtriangledown \bar{w}_{\lambda_o}(x^o) = \lim_{i \to \infty} \bigtriangledown \bar{w}_{\lambda_o+\delta_i}(x^i) = 0 \tag{8.17}$$

and

$$\bar{w}_{\lambda_o}(x^o) = \lim_{i \to \infty} \bar{w}_{\lambda_o+\delta_i}(x^i) \leq 0.$$

However, we already know $\bar{w}_{\lambda_o} \geq 0$, therefore, we must have $\bar{w}_{\lambda_o}(x^o) = 0$. It follows that

$$\bigtriangledown w_{\lambda_o}(x^o) = \bigtriangledown \bar{w}_{\lambda_o}(x^o)\phi(x^o) + \bar{w}_{\lambda_o}(x^o)\bigtriangledown\phi = 0 + 0 = 0. \tag{8.18}$$

On the other hand, by (8.14), since $w_{\lambda_o}(x^o) = 0$, x^o must be on the boundary of Σ_{λ_o}. Then by the *Hopf Lemma* (see Theorem 7.3.3), we have, the outward normal derivative

$$\frac{\partial w_{\lambda_o}}{\partial \nu}(x^o) < 0.$$

This contradicts with (8.18). Now, to verify (8.15), what left is to prove the Lemma.

Proof of Lemma 8.2.1. Assume that x^o is a negative minimum of $\bar{w}_\lambda$. Then

$$-\triangle \bar{w}_\lambda(x^o) \leq 0 \;\text{ and } \bigtriangledown \bar{w}_\lambda(x^o) = 0. \tag{8.19}$$

On the other hand, as we argued in *Step 1*, by the asymptotic behavior of u at infinity, if $|x^o|$ is sufficiently large,

$$c(x^o) := -\tau \psi^{\tau-1}(x^o) > -\frac{q(n-2-q)}{|x^o|} \equiv \frac{\triangle \phi(x^o)}{\phi(x^o)}.$$

It follows from (8.12) that

$$\left(-\triangle w_\lambda + \frac{\triangle \phi}{\phi} w_\lambda\right)(x^o) > 0.$$

This, together with (8.19) contradicts with (8.16), and hence completes the proof of the Lemma.

Step 3. In the previous two steps, we show that the positive solutions of (8.10) must be radially symmetric and monotone decreasing about some point in R^n. Since the equation is invariant under translation, without loss of generality, we may assume that the solutions are symmetric about the origin. Then they satisfies the following ordinary differential equation

$$\begin{cases} -u''(r) - \frac{n-1}{r} u'(r) = u^\tau \\ u'(0) = 0 \\ u(0) = \frac{[n(n-2)]^{(n-2)/4}}{\lambda^{\frac{n-2}{2}}} \end{cases}$$

for some $\lambda > 0$. One can verify that $u(r) = \dfrac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + r^2)^{\frac{n-2}{2}}}$ is a solution, and by the uniqueness of the ODE problem, this is the only solution. Therefore, we conclude that every positive solution of (8.10) must assume the form

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^o|^2)^{\frac{n-2}{2}}}$$

for $\lambda > 0$ and some $x^o \in R^n$. This completes the proof of the Theorem.

Proof of Theorem 8.2.3.

i) The general idea in proving this part of the Theorem is almost the same as that for Theorem 8.2.2. The main difference is that we have no decay assumption on the solution u at infinity, hence the method of moving planes can not be applied directly to u. So we first make a Kelvin transform to define a new function

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}).$$

Then obviously, $v(x)$ has the desired decay rate $\frac{1}{|x|^{n-2}}$ at infinity, but has a possible singularity at the origin. It is easy to verify that v satisfies the same equation as u does, except at the origin:

$$\triangle v + v^{\tau}(x) = 0 \quad x \in R^n \setminus \{0\}, n \geq 3. \tag{8.20}$$

We will apply the method of moving planes to the function v, and show that v is radially symmetric and monotone decreasing about some point. If the center point is the origin, then by the definition of v, u is also symmetric and monotone decreasing about the origin. If the center point of v is not the origin, then v has no singularity at the origin, and hence u has the desired decay at infinity. Then the same argument in the proof of Theorem 8.2.2 would imply that u is symmetric and monotone decreasing about some point.

Define

$$v_\lambda(x) = v(x^\lambda) \ , \quad w_\lambda(x) = v_\lambda(x) - v(x).$$

Because $v(x)$ may be singular at the origin, correspondingly w_λ may be singular at the point $x_\lambda = (2\lambda, 0, \cdots, 0)$. Hence instead of on Σ_λ, we consider w_λ on $\tilde{\Sigma}_\lambda = \Sigma_\lambda \setminus \{x_\lambda\}$. And in our proof, we treat the singular point carefully. Each time we show that the points of interest are away from the singularities, so that we can carry on the method of moving planes to the end to show the existence of a λ_o such that $w_{\lambda_o}(x) \equiv 0$ for $x \in \tilde{\Sigma}_{\lambda_o}$ and v is strictly increasing in the x_1 direction in $\tilde{\Sigma}_{\lambda_o}$.

As in the proof of Theorem 8.2.2, we see that v_λ satisfies the same equation as v does, and

$$-\triangle w_\lambda = \tau \psi_\lambda^{\tau-1}(x) w_\lambda(x).$$

where $\psi_\lambda(x)$ is some number between $v_\lambda(x)$ and $v(x)$.

Step 1. We show that, for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0, \quad \forall x \in \tilde{\Sigma}_\lambda. \tag{8.21}$$

By the asymptotic behavior

$$v(x) \sim \frac{1}{|x|^{n-2}},$$

we derive immediately that, at a negative minimum point x^o of w_λ,

$$\psi_\lambda^{\tau-1}(x^o) \sim (\frac{1}{|x^o|^{n-2}})^{\tau-1} = \frac{1}{|x^o|^4},$$

the power of $\frac{1}{|x^o|}$ is greater than two, and we have the desired decay rate for $c(x) := -\tau \psi_\lambda^{\tau-1}(x)$, as mentioned in Corollary 7.4.2. Hence we can apply the "*Decay at Infinity*" to $w_\lambda(x)$. The difference here is that, w_λ has a singularity at x_λ, hence we need to show that, the minimum of w_λ is away from x_λ. Actually, we will show that,

$$\text{If } \inf_{\tilde{\Sigma}_\lambda} w_\lambda(x) < 0, \text{ then the infimum is achieved in } \Sigma_\lambda \setminus B_1(x_\lambda) \tag{8.22}$$

To see this, we first note that for $x \in B_1(0)$,

$$v(x) \geq \min_{\partial B_1(0)} v(x) = \epsilon_0 > 0$$

due to the fact that $v(x) > 0$ and $\triangle v \leq 0$.

Then let λ be so negative, that $v(x) \leq \epsilon_0$ for $x \in B_1(x_\lambda)$. This is possible because $v(x) \to 0$ as $|x| \to \infty$.

For such λ, obviously $w_\lambda(x) \geq 0$ on $B_1(x_\lambda) \setminus \{x_\lambda\}$. This implies (8.22). Now similar to the *Step 1* in the proof of Theorem 8.2.2, we can deduce that $w_\lambda(x) \geq 0$ for λ sufficiently negative.

Step 2. Again, define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \tilde{\Sigma}_\lambda\}.$$

We will show that

$$\text{If } \lambda_o < 0, \quad \text{then } w_{\lambda_o}(x) \equiv 0, \ \forall x \in \tilde{\Sigma}_{\lambda_o}.$$

Define $\bar{w}_\lambda = \dfrac{w_\lambda}{\phi}$ the same way as in the proof of Theorem 8.2.2. Suppose that $\bar{w}_{\lambda_o}(x) \not\equiv 0$, then by *Maximum Principle* we have

$$\bar{w}_{\lambda_o}(x) > 0 \ , \quad \text{for } x \in \tilde{\Sigma}_{\lambda_o}.$$

The rest of the proof is similar to the step 2 in the proof of Theorem 8.2.2 except that now we need to take care of the singularities. Again let $\lambda_k \searrow \lambda_o$ be a sequence such that $\bar{w}_{\lambda_k}(x) < 0$ for some $x \in \tilde{\Sigma}_{\lambda_k}$. We need to show that for each k, $\inf_{\tilde{\Sigma}_{\lambda_k}} \bar{w}_{\lambda_k}(x)$ can be achieved at some point $x^k \in \tilde{\Sigma}_{\lambda_k}$ and that the sequence $\{x^k\}$ is bounded away from the singularities x_{λ_k} of w_{λ_k}. This can be seen from the following facts

a) There exists $\epsilon > 0$ and $\delta > 0$ such that

$$\bar{w}_{\lambda_o}(x) \geq \epsilon \text{ for } x \in B_\delta(x_{\lambda_o}) \setminus \{x_{\lambda_o}\}.$$

b) $\lim_{\lambda \to \lambda_o} \inf_{x \in B_\delta(x_\lambda)} \bar{w}_\lambda(x) \geq \inf_{x \in B_\delta(x_{\lambda_o})} \bar{w}_{\lambda_o}(x) \geq \epsilon.$

Fact (a) can be shown by noting that $\bar{w}_{\lambda_o}(x) > 0$ on $\tilde{\Sigma}_{\lambda_o}$ and $\triangle w_{\lambda_o} \leq 0$, while fact (b) is obvious.

Now through a similar argument as in the proof of Theorem 8.2.2 one can easily arrive at a contradiction. Therefore $\bar{w}_{\lambda_o}(x) \equiv 0$.

If $\lambda_o = 0$, then we can carry out the above procedure in the opposite direction, namely, we move the plane in the negative x_1 direction from positive infinity toward the origin. If our planes T_λ stop somewhere before the origin, we derive the symmetry and monotonicity in x_1 direction by the above argument. If they stop at the origin again, we also obtain the symmetry and monotonicity in x_1 direction by combining the two inequalities obtained in the two opposite directions. Since the x_1 direction can be chosen arbitrarily, we conclude that the solution u must be radially symmetric about some point.

ii) To show the non-existence of solutions in the case $p < \frac{n+2}{n-2}$, we notice that after the Kelvin transform, v satisfies

$$\triangle v + \frac{1}{|x|^{n+2-p(n-2)}} v^p(x) = 0 \,, \quad x \in R^n \setminus \{0\}.$$

Due to the singularity of the coefficient of v^p at the origin, one can easily see that v can only be symmetric about the origin if it is not identically zero. Hence u must also be symmetric about the origin. Now given any two points x^1 and x^2 in R^n, since equation (8.10) is invariant under translations and rotations, we may assume that the origin is at the mid point of the line segment $\overline{x^1 x^2}$. Then from the above argument, we must have $u(x^1) = u(x^2)$. It follows that u is constant. Finally, from the equation (8.10), we conclude that $u \equiv 0$. This completes the proof of the Theorem.

8.2.3 Symmetry of Solutions for $-\triangle u = e^u$ in R^2

When considering prescribing Gaussian curvature on two dimensional compact manifolds, if the sequence of approximate solutions "blows up", then by rescaling and taking limit, one would arrive at the following equation in the entire space R^2:

$$\begin{cases} \Delta u + \exp u = 0 \,, \quad x \in R^2 \\ \int_{R^2} \exp u(x) dx < +\infty \end{cases} \tag{8.23}$$

The classification of the solutions for this limiting equation would provide essential information on the original problems on manifolds, also it is interesting in its own right.

It is known that

$$\phi_{\lambda, x^o}(x) = \ln \frac{32\lambda^2}{(4 + \lambda^2 |x - x^o|^2)^2}$$

for any $\lambda > 0$ and any point $x^o \in R^2$ is a family of explicit solutions.

We will use the method of moving planes to prove:

Theorem 8.2.4 *Every solution of (8.23) is radially symmetric with respect to some point in R^2 and hence assumes the form of $\phi_{\lambda, x^o}(x)$.*

To this end, we first need to obtain some decay rate of the solutions near infinity.

Some Global and Asymptotic Behavior of the Solutions

The following Theorem gives the asymptotic behavior of the solutions near infinity, which is essential to the application of the method of moving planes.

Theorem 8.2.5 *If $u(x)$ is a solution of (8.23), then as $|x| \to +\infty$,*

$$\frac{u(x)}{\ln |x|} \to -\frac{1}{2\pi} \int_{R^2} \exp u(x) dx \leq -4$$

uniformly.

This Theorem is a direct consequence of the following two Lemmas.

Lemma 8.2.2 *(W. Ding) If u is a solution of*

$$-\triangle u = e^u, \quad x \in R^2$$

and

$$\int_{R^2} \exp u(x) dx < +\infty,$$

then

$$\int_{R^2} \exp u(x) dx \geq 8\pi.$$

Proof. For $-\infty < t < \infty$, let $\Omega_t = \{x \mid u(x) > t\}$, one can obtain

$$\int_{\Omega_t} \exp u(x) dx = -\int_{\Omega_t} \triangle u = \int_{\partial \Omega_t} |\triangledown u| ds$$

$$-\frac{d}{dt} |\Omega_t| = \int_{\partial \Omega_t} \frac{ds}{|\triangledown u|}$$

By the Schwartz inequality and the isoperimetric inequality,

$$\int_{\partial \Omega_t} \frac{ds}{|\triangledown u|} \cdot \int_{\partial \Omega_t} |\triangledown u| \geq |\partial \Omega_t|^2 \geq 4\pi |\Omega_t|.$$

Hence

$$-(\frac{d}{dt} |\Omega_t|) \cdot \int_{\Omega_t} \exp u(x) dx \geq 4\pi |\Omega_t|$$

and so

$$\frac{d}{dt} (\int_{\Omega_t} \exp u(x) dx)^2 = 2 \exp t \cdot (\frac{d}{dt} |\Omega_t|) \cdot \int_{\Omega_t} \exp u(x) dx \leq -8\pi |\Omega_t| e^t.$$

Integrating from $-\infty$ to ∞ gives

$$-(\int_{R^2} \exp u(x) dx)^2 \leq -8\pi \int_{R^2} \exp u(x) dx$$

which implies $\int_{R^2} \exp u(x) dx \geq 8\pi$ as desired.

Lemma 8.2.2 enables us to obtain the asymptotic behavior of the solutions at infinity.

Lemma 8.2.3 . *If $u(x)$ is a solution of (8.23), then as $|x| \to +\infty$,*

$$\frac{u(x)}{\ln|x|} \to -\frac{1}{2\pi}\int_{R^2} \exp u(x)dx \text{ uniformly.}$$

Proof.

By a result of Brezis and Merle [BM], we see that the condition $\int_{R^2} \exp u(x)dx < \infty$ implies that the solution u is bounded from above.

Let

$$w(x) = \frac{1}{2\pi}\int_{R^2} (\ln|x-y| - \ln(|y|+1)) \exp u(y)dy.$$

Then it is easy to see that

$$\triangle w(x) = \exp u(x) \ , \quad x \in R^2$$

and we will show

$$\frac{w(x)}{\ln|x|} \to \frac{1}{2\pi}\int_{R^2} \exp u(x)dx \text{ uniformly as } |x| \to +\infty. \tag{8.24}$$

To see this, we need only to verify that

$$I := \int_{R^2} \frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|} e^{u(y)}dy {\rightarrow} 0$$

as $|x|{\rightarrow}\infty$. Write $I = I_1 + I_2 + I_3$, where I_1, I_2 and I_3 are the integrals on the three regions

$$D_1 = \{y \mid |x-y| \leq 1\},$$

$$D_2 = \{y \mid |x-y| > 1 \text{ and } |y| \leq K\}$$

and

$$D_3 = \{y \mid |x-y| > 1 \text{ and } |y| > K\}$$

respectively. We may assume that $|x| \geq 3$.

a) To estimate I_1, we simply notice that

$$I_1 \leq C\int_{|x-y|\leq 1} e^{u(y)}dy - \frac{1}{\ln|x|}\int_{|x-y|\leq 1} \ln|x-y| e^{u(y)}dy$$

Then by the boundedness of $e^{u(y)}$ and $\int_{R^2} e^{u(y)}dy$, we see that $I_1{\rightarrow}0$ as $|x|{\rightarrow}\infty$.

b) For each fixed K, in region D_2, we have, as $|x|{\rightarrow}\infty$,

$$\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|} {\rightarrow} 0$$

hence $I_2{\rightarrow}0$.

c)To see $I_3 \to 0$, we use the fact that for $|x-y|>1$

$$|\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|}| \leq C$$

Then let $K \to \infty$. This verifies (8.24).

Consider the function $v(x) = u(x) + w(x)$. Then $\triangle v \equiv 0$ and

$$v(x) \leq C + C_1 \ln(|x|+1)$$

for some constant C and C_1. Therefore v must be a constant. This completes the proof of our Lemma.

Combining Lemma 8.2.2 and Lemma 8.2.3, we obtain our Theorem 8.2.5.

The Method of Moving Planes and Symmetry

In this subsection, we apply the method of moving planes to establish the symmetry of the solutions. For a given solution, we move the family of lines which are orthogonal to a given direction from negative infinity to a critical position and then show that the solution is symmetric in that direction about the critical position. We also show that the solution is strictly increasing before the critical position. Since the direction can be chosen arbitrarily, we conclude that the solution must be radially symmetric about some point. Finally by the uniqueness of the solution of the following O.D.E. problem

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) = f(u) \\ u'(0) = 0 \\ u(0) = 1 \end{cases}$$

we see that the solutions of (8.23) must assume the form $\phi_{\lambda,x^0}(x)$.

Assume that $u(x)$ is a solution of (8.23). Without loss of generality, we show the monotonicity and symmetry of the solution in the x_1 direction.

For $\lambda \in R^1$, let

$$\Sigma_\lambda = \{(x_1, x_2) \mid x_1 < \lambda\}$$

and

$$T_\lambda = \partial \Sigma_\lambda = \{(x_1, x_2) \mid x_1 = \lambda\}.$$

Let

$$x^\lambda = (2\lambda - x_1, x_2)$$

be the reflection point of $x = (x_1, x_2)$ about the line T_λ. (See the previous Figure 1.)

Define

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

A straight forward calculation shows

$$\triangle w_\lambda(x) + (\exp \psi_\lambda(x)) w_\lambda(x) = 0 \tag{8.25}$$

where $\psi_\lambda(x)$ is some number between $u(x)$ and $u_\lambda(x)$.

Step 1. As in the previous examples, we show that, for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0, \;\; \forall x \in \Sigma_\lambda.$$

By the "*Decay at Infinity*" argument, the key is to check the decay rate of $c(x) := -e^{\psi_\lambda(x)}$ at points x^o where $w_\lambda(x)$ is negative. At such point

$$u(x^{o\lambda}) < u(x^o), \;\; \text{and hence } \psi_\lambda(x^o) \leq u(x^o).$$

By virtue of Theorem 8.2.5, we have

$$e^{\psi_\lambda(x^o)} = O(\frac{1}{|x^o|^4}) \tag{8.26}$$

Notice that we are in dimension two, while the key function $\phi = \frac{1}{|x|^q}$ given in Corollary 7.4.2 requires $0 < q < n-2$, hence it does not work here. As a modification, we choose

$$\phi(x) = \ln(|x| - 1).$$

Then it is easy to verify that

$$\frac{\triangle \phi}{\phi}(x) = \frac{-1}{|x|(|x|-1)^2 \ln(|x|-1)}.$$

It follows from this and (8.26) that

$$e^{\psi(x^o)} + \frac{\triangle \phi}{\phi}(x^o) < 0 \quad \text{for sufficiently large } |x^o|. \tag{8.27}$$

This is what we desire for.

Then similar to the argument in Subsection 5.2, we introduce the function

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)}.$$

It satisfies

$$\triangle \bar{w}_\lambda + 2 \triangledown \bar{w}_\lambda \frac{\triangledown \phi}{\phi} + \left(e^{\psi_\lambda(x)} + \frac{\triangle \phi}{\phi} \right) \bar{w}_\lambda = 0. \tag{8.28}$$

Moreover, by the asymptotic behavior of the solution u near infinity (see Lemma 8.2.3), we have, for each fixed λ,

$$\bar{w}_\lambda(x) = \frac{u(x^\lambda) - u(x)}{\ln(|x|-1)} \rightarrow 0\,, \quad \text{as } |x| \rightarrow \infty. \tag{8.29}$$

Now, if $w_\lambda(x) < 0$ somewhere, then by (8.29), there exists a point x^o, which is a negative minimum of $\bar{w}_\lambda(x)$. At this point, one can easily derive a contradiction from (8.27) and (8.28). This completes *Step 1.*

Step 2 . Define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\}.$$

We show that

$$\text{If } \lambda_o < 0, \ \text{ then } w_{\lambda_o}(x) \equiv 0, \ \forall x \in \Sigma_{\lambda_o}.$$

The argument is entirely similar to that in the *Step 2* of the proof of Theorem 8.2.2 except that we use $\phi(x) = \ln(-x_1 + 2)$. This completes the proof of the Theorem.

8.3 Method of Moving Planes in a Local Way

8.3.1 The Background

Let M be a Riemannian manifold of dimension $n \geq 3$ with metric g_o. Given a function $R(x)$ on M, one interesting problem in differential geometry is to find a metric g that is point-wise conformal to the original metric g_o and has scalar curvature R(x). This is equivalent to finding a positive solution of the semi-linear elliptic equation

$$-\frac{4(n-1)}{n-2} \triangle_o u + R_o(x) u = R(x) u^{\frac{n+2}{n-2}} , \ \ x \in M, \tag{8.30}$$

where $\triangle_o$ is the Beltrami-Laplace operator of (M, g_o) and $R_o(x)$ is the scalar curvature of g_o.

In recent years, there have seen a lot of progress in understanding equation (8.30). When (M, g_o) is the standard sphere S^n, the equation becomes

$$-\triangle u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} R(x) u^{\frac{n+2}{n-2}}, \ \ u > 0, \ x \in S^n. \tag{8.31}$$

It is the so-called critical case where the lack of compactness occurs. In this case, the well-known Kazdan-Warner condition gives rise to many examples of R in which there is no solution. In the last few years, a tremendous amount of effort has been put to find the existence of the solutions and many interesting results have been obtained (see [Ba] [BC] [Bi] [BE] [CL1] [CL3] [CL4] [CL6] [CY1] [CY2] [ES] [KW1] [Li2] [Li3] [SZ] [Lin1] and the references therein).

One main ingredient in the proof of existence is to obtain a priori estimates on the solutions. For equation (8.31) on S^n, to establish a priori estimates, a useful technique employed by most authors was a 'blowing-up' analysis. However, it does not work near the points where $R(x) = 0$. Due to this limitation, people had to assume that R was positive and bounded away from 0. This technical assumption became somewhat standard and has been used by many authors for quite a few years. For example, see articles by Bahri

[Ba], Bahri and Coron [BC], Chang and Yang [CY1], Chang, Gursky, and Yang [CY2], Li [Li2] [Li3], and Schoen and Zhang [SZ].

Then for functions with changing signs, are there any a priori estimates?

In [CL6], we answered this question affirmatively. We removed the above well-known technical assumption and obtained a priori estimates on the solutions of (8.31) for functions R which are allowed to change signs.

The main difficulty encountered by the traditional 'blowing-up' analysis is near the point where $R(x) = 0$. To overcome this, we used the 'method of moving planes' in a local way to control the growth of the solutions in this region and obtained an a priori estimate.

In fact, we derived a priori bounds for the solutions of more general equations

$$-\triangle u + \frac{n(n-2)}{4} u = R(x) u^p, \quad u > 0, \; x \in S^n. \tag{8.32}$$

for any exponent p between $1 + \frac{1}{A}$ and A, where A is an arbitrary positive number. For a fixed A, the bound is independent of p.

Proposition 8.3.1 *Assume $R \in C^{2,\alpha}(S^n)$ and $|\bigtriangledown R(x)| \geq \beta_0$ for any x with $|R(x)| \leq \delta_0$. Then there are positive constants ϵ and C depending only on β_0, δ_0, M, and $\|R\|_{C^{2,\alpha}(S^n)}$, such that for any solution u of (8.32), we have $u \leq C$ in the region where $|R(x)| \leq \epsilon$.*

In this proposition, we essentially required $R(x)$ to grow linearly near its zeros, which seems a little restrictive. Recently, in the case $p = \frac{n+2}{n-2}$, Lin [Lin1] weaken the condition by assuming only polynomial growth of R near its zeros. To prove this result, he first established a Liouville type theorem in R^n, then use a blowing up argument.

Later, by introducing a new auxiliary function, we sharpen our method of moving planes to further weaken the condition and to obtain more general result:

Theorem 8.3.1 *Let M be a locally conformally flat manifold. Let*

$$\Gamma = \{x \in M \mid R(x) = 0\}.$$

Assume that $\Gamma \in C^{2,\alpha}$ and $R \in C^{2,\alpha}(M)$ satisfying $\dfrac{\partial R}{\partial \nu} \leq 0$ where ν is the outward normal (pointing to the region where R is negative) of Γ, and

$$\frac{R(x)}{|\bigtriangledown R(x)|} \quad \text{is} \quad \begin{cases} \text{continuous near } \Gamma, \\ = 0 \text{ at } \Gamma. \end{cases} \tag{8.33}$$

Let D be any compact subset of M and let p be any number greater than 1. Then the solutions of the equation

$$-\triangle_o u + R_o(x) u = R(x) u^p, \quad \text{in } M \tag{8.34}$$

are uniformly bounded near $\Gamma \cap D$.

One can see that our condition (8.33) is weaker than Lin's polynomial growth restriction. To illustrate, one may simply look at functions R(x) which grow like $\exp\{-\frac{1}{d(x)}\}$, where d(x) is the distance from the point x to Γ. Obviously, this kinds of functions satisfy our condition, but are not polynomial growth.

Moreover our restriction on the exponent is much weaker. Besides being $\frac{n+2}{n-2}$, p can be any number greater than 1.

8.3.2 The A Priori Estimates

Now, we estimate the solutions of equation (8.34) and prove Theorem 8.3.1.

Since M is a locally conformally flat manifold, a local flat metric can be chosen so that equation (8.34) in a compact set $D \subset M$ can be reduced to

$$-\triangle u = R(x)u^p, \;\; p > 1, \tag{8.35}$$

in a compact set K in R^n. This is the equation we will consider throughout the section.

The proof of the Theorem is divided into two parts. We first derive the bound in the region(s) where R is negatively bounded away from zero. Then based on this bound, we use the 'method of moving planes' to estimate the solutions in the region(s) surrounding the set Γ.

Part I. In the region(s) where R is negative

In this part, we derive estimates in the region(s) where R is negatively bounded away from zero. It comes from a standard elliptic estimate for subharmonic functions and an integral bound on the solutions. We prove

Theorem 8.3.2 *The solutions of (8.35) are uniformly bounded in the region* $\{x \in K : R(x) \leq 0$ *and* $dist(x, \Gamma) \geq \delta\}$*. The bound depends only on* δ*, the lower bound of* R*.*

To prove Theorem 8.3.2, we introduce the following lemma of Gilbarg and Trudinger [GT].

Lemma 8.3.1 *(See Theorem 9.20 in [GT]). Let* $u \in W^{2,n}(D)$ *and suppose* $\triangle u \geq 0$*. Then for any ball* $B_{2r}(x) \subset D$ *and any* $q > 0$*, we have*

$$\sup_{B_r(x)} u \leq C(\frac{1}{r^n}\int_{B_{2r}(x)} (u^+)^q dV)^{\frac{1}{q}}, \tag{8.36}$$

where $C = C(n, D, q)$*.*

In virtue of this Lemma, to establish an a priori bound of the solutions, one only need to obtain an integral bound. In fact, we prove

Lemma 8.3.2 *Let B be any ball in K that is bounded away from Γ, then there is a constant C such that*

$$\int_B u^p dx \leq C. \tag{8.37}$$

Proof. Let Ω be an open neighborhood of K such that dist $(\partial\Omega, \text{K}) \geq 1$. Let φ be the first eigenfunction of $-\triangle$ in Ω, i.e.

$$\begin{cases} -\triangle\varphi = \lambda_1\varphi(x) & x \in \Omega \\ \varphi(x) = 0 & x \in \partial\Omega. \end{cases}$$

Let

$$sgnR = \begin{cases} 1 & \text{if } R(x) > 0 \\ 0 & \text{if } R(x) = 0 \\ -1 & \text{if } R(x) < 0. \end{cases}$$

Let $\alpha = \frac{1+2p}{p-1}$. Multiply both sides of equation (8.35) by $\varphi^\alpha|R|^\alpha sgnR$ and then integrate. Taking into account that $\alpha > 2$ and φ and R are bounded in Ω, through a straight forward calculation, we have

$$\int_\Omega \varphi^\alpha|R|^{1+\alpha}u^p dx \leq C_1 \int_\Omega \varphi^{\alpha-2}|R|^{\alpha-2}udx \leq C_2 \int_\Omega \varphi^{\frac{\alpha}{p}}|R|^{\alpha-2}udx.$$

Applying Hölder inequality on the right-hand-side, we arrive at

$$\int_\Omega \varphi^\alpha|R|^{1+\alpha}u^p dx \leq C_3. \tag{8.38}$$

Now (8.37) follows suit. This completes the proof of the Lemma.

The Proof of Theorem 8.3.2. It is a direct consequence of Lemma 8.3.1 and Lemma 8.3.2

Part II. In the region where R is small

In this part, we estimate the solutions in a neighborhood of Γ, where R is small.

Let u be a given solution of (8.35), and $x_0 \in \Gamma$. It is sufficient to show that u is bounded in a neighborhood of x_0. The key ingredient is to use the method of moving planes to show that, near x_0, along any direction that is close to the direction of $\bigtriangledown R$, the values of the solution u is comparable. This result, together with the bounded-ness of the integral $\int u^p$ in a small neighborhood of x_0, leads to an a priori bound of the solutions. Since the method of moving planes can not be applied directly to u, we construct some auxiliary functions. The proof consists of the following four steps.

Step 1. Transforming the region

Make a translation, a rotation, or if necessary, a Kelvin transform. Denote the new system by $x = (x_1, y)$ with $y = (x_2, \cdots, x_n)$. Let x_1 axis pointing to the right. In the new system, x_0 becomes the origin, the image of the set $\Omega^+ := \{x \mid R(x) > 0\}$ is contained in the left of the y-plane, and the image of Γ is tangent to the y-plane at origin. The image of Γ is also uniformly convex near the origin.

Let $x_1 = \phi(y)$ be the equation of the image of Γ near the origin. Let

$$\partial_1 D := \{x \mid x_1 = \phi(y) + \epsilon, x_1 \geq -2\epsilon\}$$

The intersection of $\partial_1 D$ and the plane $\{x \in R^n \mid x_1 = -2\epsilon\}$ encloses a part of the plane, which is denote by $\partial_2 D$. Let D by the region enclosed by two surfaces $\partial_1 D$ and $\partial_2 D$. (See Figure 6.)

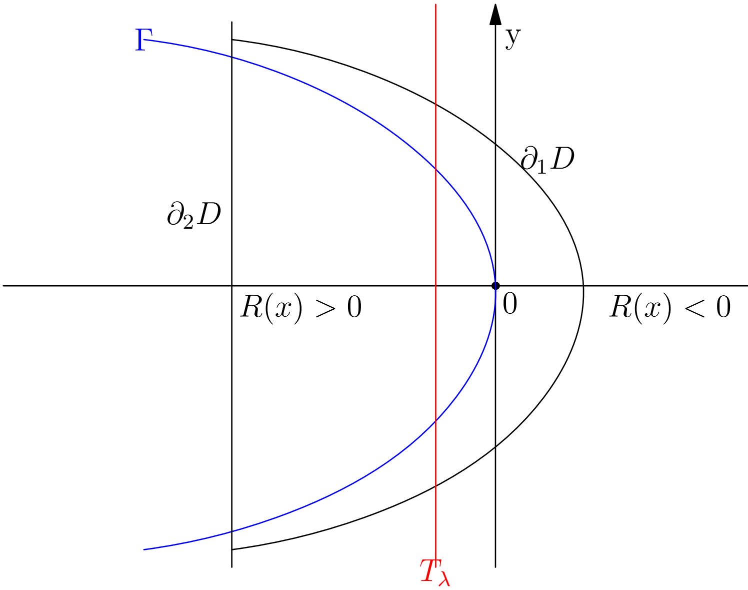


Figure 6

The small positive number ϵ is chosen to ensure that

(a) $\partial_1 D$ is uniformly convex,
(b) $\frac{\partial R}{\partial x_1} \leq 0$ in D, and
(c) $\dfrac{R(x)}{|\bigtriangledown R|}$ is continuous in D.

Step 2. Introducing an auxiliary function

Let $m = \max_{\partial_1 D} u$. Let $\tilde{u}$ be a C^2 extension of u from $\partial_1 D$ to the entire ∂D, such that

$$0 \leq \tilde{u} \leq 2m, \quad \text{and } |\bigtriangledown \tilde{u}| \leq Cm.$$

Let w be the harmonic function

$$\begin{cases} \triangle w = 0 & x \in D \\ w = \tilde{u} & x \in \partial D \end{cases}$$

Then by the maximum principle and a standard elliptic estimate, we have

$$0 \leq w(x) \leq 2m \quad \text{and } \|w\|_{C^1(D)} \leq C_1 m. \tag{8.39}$$

Introduce a new function

$$v(x) = u(x) - w(x) + C_0 m[\epsilon + \phi(y) - x_1] + m[\epsilon + \phi(y) - x_1]^2$$

which is well defined in D. Here C_0 is a large constant to be determined later. One can easily verify that $v(x)$ satisfies the following

$$\begin{cases} \triangle v + \psi(y) + f(x, v) = 0 & x \in D \\ v(x) = 0 & x \in \partial_1 D \end{cases}$$

where

$$\psi(y) = -C_0 m \triangle \phi(y) - m \triangle [\epsilon + \phi(y)]^2 - 2m,$$

and

$$f(x, v) = 2m x_1 \triangle \phi(y) + R(x)\{v + w(x) - C_0 m[\epsilon + \phi(y) - x_1] - m[\epsilon + \phi(y) - x_1]^2\}^p$$

At this step, we show that

$$v(x) > 0 \quad \forall x \in D.$$

We consider the following two cases.

Case 1) $\phi(y) + \frac{\epsilon}{2} \leq x_1 \leq \phi(y) + \epsilon$. In this region, $R(x)$ is negative and bounded away from 0. By Theorem 8.3.2 and the standard elliptic estimate, we have

$$\left|\frac{\partial u}{\partial x_1}\right| \leq C_2 m,$$

for some positive constant C_2. It follows from this and (8.39),

$$\frac{\partial v}{\partial x_1} \leq (C_1 + C_2 - C_0) m - 2m(\epsilon + \phi(y) - x_1).$$

Noticing that $(\epsilon + \phi(y) - x_1) \geq 0$, one can choose C_0 sufficiently large, such that

$$\frac{\partial v}{\partial x_1} < 0$$

Then since

$$v(x) = 0 \quad \forall x \in \partial_1 D$$

we have $v(x) > 0$.

Case 2) $-2\epsilon \leq x_1 \leq \phi(y) + \frac{\epsilon}{2}$. In this region, again by (8.39), we have

$$v(x) \geq -w(x) + C_0 m \frac{\epsilon}{2} \geq -2m + C_0 m \frac{\epsilon}{2}.$$

Again, choosing C_0 sufficiently large (depending only on ϵ), we arrive at $v(x) > 0$.

Step 3. Applying the Method of Moving Planes to v in x_1 direction

In the previous step, we showed that $v(x) \geq 0$ in D and $v(x) = 0$ on $\partial_1 D$. This lies a foundation for the moving of planes. Now we can start from the rightmost tip of region D, and move the plane perpendicular to x_1 axis toward the left. More precisely, let

$$\Sigma_\lambda = \{x \in D \mid x_1 \geq \lambda\},$$

$$T_\lambda = \{x \in R^n \mid x_1 = \lambda\}.$$

Let $x^\lambda = (2\lambda - x_1, y)$ be the reflection point of x with respect to the plane T_λ. Let

$$v_\lambda(x) = v(x^\lambda) \quad \text{and } w_\lambda(x) = v_\lambda(x) - v(x).$$

We are going to show that

$$w_\lambda(x) \geq 0 \;\; \forall\, x \in \Sigma_\lambda. \tag{8.40}$$

We want to show that the above inequality holds for $-\epsilon_1 \leq \lambda \leq \epsilon$ for some $0 < \epsilon_1 < \epsilon$. This choice of ϵ_1 is to guarantee that when the plane T_λ moves to $\lambda = -\epsilon_1$, the reflection of Σ_λ .about the plane T_λ still lies in region D.

(8.40) is true when λ is less but close to ϵ, because Σ_λ is a narrow region and $f(x,v)$ is Lipschitz continuous in v (Actually, $\frac{\partial f}{\partial v} = R(x)$). For detailed argument, the readers may see the first example in Section 5.

Now we decrease λ, that is, move the plane T_λ toward the left as long as the inequality (8.40) remains true. We show that the moving of planes can be carried on provided

$$f(x, v(x)) \leq f(x^\lambda, v(x)) \text{ for } x = (x_1, y) \in D \text{ with } x_1 > \lambda > -\epsilon_1. \tag{8.41}$$

In fact, it is easy to see that v_λ satisfies the equation

$$\triangle v_\lambda + \psi(y) + f(x^\lambda, v_\lambda) = 0,$$

and hence w_λ satisfies

$$\begin{aligned} -\triangle w_\lambda &= f(x^\lambda, v_\lambda) - f(x^\lambda, v) + f(x^\lambda, v) - f(x, v) \\ &= R(x^\lambda)(v_\lambda - v) + f(x^\lambda, v) - f(x, v) \\ &= R(x^\lambda) w_\lambda + f(x^\lambda, v) - f(x, v). \end{aligned}$$

If (8.41) holds, then we have

$$-\triangle w_\lambda - R(x^\lambda) w_\lambda(x) \geq 0. \tag{8.42}$$

This enables us to apply the maximum principle to w_λ.

Let

$$\lambda_o = \inf\{\lambda \mid w_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda\}.$$

If $\lambda_o > -\epsilon_1$, then since $v(x) > 0$ in D, by virtue of (8.42) and the maximum principle, we have

$$w_{\lambda_o}(x) > 0, \ \forall x \in \Sigma_{\lambda_o} \quad \text{and} \quad \frac{\partial w_{\lambda_o}}{\partial x_1} < 0, \ \forall x \in T_{\lambda_o} \cap \Sigma_{\lambda_o}.$$

Then, similar to the argument in the examples in Section 5, one can derive a contradiction.

It is elementary to see that (8.41) holds if $\dfrac{\partial f(x,v)}{\partial x_1} \leq 0$ in the set

$$\{x \in D \mid R(x) < 0 \text{ or } x_1 > -2\epsilon_1\}.$$

To estimate $\dfrac{\partial f}{\partial x_1}$, we use

$$\frac{\partial f}{\partial x_1} = 2m\triangle\phi(y) + u^{p-1}\{\frac{\partial R}{\partial x_1} u + R(x) p [\frac{\partial w}{\partial x_1} + C_0 m + 2m(\epsilon + \phi(y) - x_1)]\}.$$

First, we notice that the uniform convexity of the image of Γ near the origin implies

$$\triangle\phi(y) \leq -a_0 < 0. \tag{8.43}$$

for some constant a_0.

We consider the following two possibilities.

a) $R(x) \leq 0$. Choose C_0 sufficiently large, so that

$$\frac{\partial w}{\partial x_1} + C_0 m \geq 0$$

Noticing that $\frac{\partial R}{\partial x_1} \leq 0$ and $(\epsilon + \phi(y) - x_1) \geq 0$ in D, we have

$$\frac{\partial f}{\partial x_1} \leq 0.$$

b) $R(x) > 0$, $x = (x_1, y)$ with $x_1 > -2\epsilon_1$.

In the part where $u \geq 1$, we use u to control

$$\left(R(x) / \frac{\partial R}{\partial x_1} \right) p [\frac{\partial w}{\partial x_1} + C_0 m + 2m(\epsilon + \phi(y) - x_1)].$$

More precisely, we write

$$\frac{\partial f}{\partial x_1} = 2m\triangle\phi(y) + u^{p-1} \frac{\partial R}{\partial x_1} \left\{ u + \left(R(x) / \frac{\partial R}{\partial x_1} \right) p [\frac{\partial w}{\partial x_1} + C_0 m + 2m(\epsilon + \phi(y) - x_1)] \right\}.$$

Choose ϵ_1 sufficiently small, such that

$$|\frac{\partial R}{\partial x_1}| \sim |\bigtriangledown R|.$$

Then by condition (8.33) on R, we can make $R(x)/\frac{\partial R}{\partial x_1}$ arbitrarily small by a small choice of ϵ_1, and therefore arrive at $\frac{\partial f}{\partial x_1} \leq 0$.

In the part where $u \leq 1$, in virtue of (8.43), we can use $2m\triangle\phi(y)$ to control

$$Rp[\frac{\partial w}{\partial x_1} + C_0 m + 2m(\epsilon + \phi(y) - x_1)].$$

Here again we use the smallness of R.

So far, our conclusion is: The method of moving planes can be carried on up to $\lambda = -\epsilon_1$. More precisely, for any λ between $-\epsilon_1$ and ϵ, the inequality (8.40) is true.

Step 4. Deriving the a priori bound

Inequality (8.40) implies that, in a small neighborhood of the origin, the function $v(x)$ is monotone decreasing in x_1 direction. A similar argument will show that this remains true if we rotate the x_1-axis by a small angle. Therefore, for any point $x_0 \in \Gamma$, one can find a cone Δ_{x_0} with x_0 as its vertex and staying to the left of x_0, such that

$$v(x) \geq v(x_0) \quad \forall x \in \Delta_{x_0} \tag{8.44}$$

Noticing that $w(x)$ is bounded in D, (8.44) leads immediately to

$$u(x) + C_6 \geq u(x_0) \quad \forall x \in \Delta_{x_0} \tag{8.45}$$

More generally, by a similar argument, one can show that (8.45) is true for any point x_0 in a small neighborhood of Γ. Furthermore, the intersection of the cone Δ_{x_0} with the set $\{x | R(x) \geq \frac{\delta_0}{2}\}$ has a positive measure and the lower bound of the measure depends only on δ_0 and the C^1 norm of R. Now the a priori bound of the solutions is a consequence of (8.45) and an integral bound on u, which can be derived from Lemma 8.3.2.

This completes the proof of Theorem 8.3.1.

8.4 Method of Moving Spheres

8.4.1 The Background

Given a function $K(x)$ on the two dimensional standard sphere S^2, the well-known Nirenberg problem is to find conditions on $K(x)$, so that it can be realized as the Gaussian curvature of some conformally related metric. This is equivalent to solving the following nonlinear elliptic equation

$$-\triangle u + 1 = K(x)e^{2u} \tag{8.46}$$

on S^2. Where $\triangle$ is the Laplacian operator associated to the standard metric.

On the higher dimensional sphere S^n, a similar problem was raised by Kazdan and Warner:

Which functions $R(x)$ can be realized as the scalar curvature of some conformally related metrics ?

The problem is equivalent to the existence of positive solutions of another elliptic equation

$$-\triangle u + \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}} \tag{8.47}$$

For a unifying description, on S^2, we let $R(x) = 2K(x)$ be the scalar curvature, then (8.46) is equivalent to

$$-\triangle u + 2 = R(x)e^u. \tag{8.48}$$

Both equations are so-called 'critical' in the sense that lack of compactness occurs. Besides the obvious necessary condition that $R(x)$ be positive somewhere, there are other well-known obstructions found by Kazdan and Warner [KW1] and later generalized by Bourguignon and Ezin [BE]. The conditions are:

$$\int_{S^n} X(R)dA_g = 0 \tag{8.49}$$

where dA_g is the volume element of the metric $e^u g_o$ and $u^{\frac{4}{n-2}} g_o$ for $n = 2$ and for $n \geq 3$ respectively. The vector field X in (8.49) is a conformal Killing field associated to the standard metric g_o on S^n. In the following, we call these necessary conditions Kazdan-Warner type conditions.

These conditions give rise to many examples of $R(x)$ for which (8.48) and (8.47) have no solution. In particular, a monotone rotationally symmetric function R admits no solution.

Then for which R, can one solve the equations? What are the necessary and sufficient conditions for the equations to have a solution? These are very interesting problems in geometry.

In recent years, various sufficient conditions were obtained by many authors (For example, see [BC] [CY1] [CY2] [CY3] [CY4] [CL10] [CD1] [CD2] [CC] [CS] [ES] [Ha] [Ho] [KW1] [Kw2] [Li1] [Li2] [Mo] and the references therein.) However, there are still gaps between those sufficient conditions and the necessary ones. Then one may naturally ask: ' Are the necessary conditions of Kazdan-Warner type also sufficient?' This question has been open for many years. In [CL3], we gave a negative answer to this question.

To find necessary and sufficient conditions, it is natural to begin with functions having certain symmetries. A pioneer work in this direction is [Mo] by Moser. He showed that, for even functions R on S^2, the necessary and sufficient condition for (8.48) to have a solution is R being positive somewhere.

Note that, in the class of even functions, (1.2) is satisfied automatically. Thus one can interpret Moser's result as:

In the class of even functions, the Kazdan-Warner type conditions are the necessary and sufficient conditions for (8.48) to have a solution.

A simple question was asked by Kazdan [Ka]:

What are the necessary and sufficient conditions for rotationally symmetric functions?

To be more precise, for rotationally symmetric functions $R = R(\theta)$ where θ is the distance from the north pole, the Kazdan-Warner type conditions take the form:

$$R > 0 \text{ somewhere and } R' \text{ changes signs} \tag{8.50}$$

In [XY], Xu and Yang found a family of rotationally symmetric functions R_ϵ satisfying conditions (8.50), for which problem (8.48) has no rotationally symmetric solution. Even though one didn't know whether there were any other non-symmetric solutions for these functions R_ϵ, this result is still very interesting, because it suggests that (8.50) may not be the sufficient condition.

In our earlier paper [CL3], we present a class of rotationally symmetric functions satisfying (8.50) for which problem (8.48) has no solution at all. And thus, for the first time, pointed out that the Kazdan- Warner type conditions are not sufficient for (8.48) to have a solution. First, we proved the non-existence of rotationally symmetric solutions:

Proposition 8.4.1 *Let R be rotationally symmetric. If*

$$R \text{ is monotone in the region where } R > 0, \text{ and } R \not\equiv C, \tag{8.51}$$

Then problem (8.48) and (8.47) has no rotationally symmetric solution.

This generalizes Xu and Yang's result, since their family of functions R_ϵ satisfy (8.51). We also cover the higher dimensional cases.

Although we believed that for all such functions R, there is no solution at all, we were not able to prove it at that time. However, we could show this for a family of such functions.

Proposition 8.4.2 *There exists a family of functions R satisfying the Kazdan-Warner type conditions (8.50), for which the problem (8.48) has no solution at all.*

At that time, we were not able to obtain the counter part of this result in higher dimensions.

In [XY], Xu and Yang also proved:

Proposition 8.4.3 *Let R be rotationally symmetric. Assume that*
i) R is non-degenerate
ii)

$$R' \text{ changes signs in the region where } R > 0 \tag{8.52}$$

Then problem (8.48) has a solution.

The above results and many other existence results tend to lead people to believe that what really counts is whether R' changes signs in the region where $R > 0$. And we conjectured in [CL3] that for rotationally symmetric R, condition (8.52), instead of (8.50), would be the necessary and sufficient condition for (8.48) or (8.47) to have a solution.

The main objective of this section is to present the proof on the necessary part of the conjecture, which appeared in our later paper [CL4]. In [CL3], to prove Proposition 8.4.2, we used the method of 'moving planes' to show that the solutions of (8.48) are rotationally symmetric. This method only work for a special family of functions satisfying (8.51). It does not work in higher dimensions. In [CL4], we introduce a new idea. We call it the method of 'moving spheres'. It works in all dimensions, and for all functions satisfying (8.51). Instead of showing the symmetry of the solutions, we obtain a comparison formula which leads to a direct contradiction. In an earlier work [P] , a similar method was used by P. Padilla to show the radial symmetry of the solutions for some nonlinear Dirichlet problems on annuluses.

The following are our main results.

Theorem 8.4.1 *Let R be continuous and rotationally symmetric. If*

$$R \text{ is monotone in the region where } R > 0, \text{ and } R \not\equiv C, \tag{8.53}$$

Then problems (8.48) and (8.47) have no solution at all.

We also generalize this result to a family of non-symmetric functions.

Theorem 8.4.2 *Let R be a continuous function. Let B be a geodesic ball centered at the North Pole $x_{n+1} = 1$. Assume that*
i) $R(x) \geq 0$, R is non-decreasing in x_{n+1} direction, for $x \in B$;
ii) $R(x) \leq 0$, for $x \in S^n \setminus B$;
Then problems (8.48) and (8.47) have no solution at all.

Combining our Theorem 1 with Xu and Yang's Proposition 3, one can obtain a necessary and sufficient condition in the non-degenerate case:

Corollary 8.4.1 *Let R be rotationally symmetric and non-degenerate in the sense that $R'' \neq 0$ whenever $R' = 0$. Then the necessary and sufficient condition for (8.48) to have a solution is*

$$R > 0 \text{ somewhere and } R' \text{ changes signs in the region where } R > 0$$

8.4.2 Necessary Conditions

In this subsection, we prove Theorem 8.4.1 and Theorem 8.4.2.

For convenience, we make a stereo-graphic projection from S^n to R^n. Then it is equivalent to consider the following equation in Euclidean space R^n :

$$-\triangle u = R(r)e^u,\ x \in R^2. \tag{8.54}$$

$$-\triangle u = R(r)u^\tau\ u > 0,\ x \in R^n,\ n \geq 3, \tag{8.55}$$

with appropriate asymptotic growth of the solutions at infinity

$$u \sim -4\ln|x| \text{ for n=2 }; \quad u \sim C|x|^{2-n} \text{ for } n \geq 3 \tag{8.56}$$

for some $C > 0$. Here $\triangle$ is the Euclidean Laplacian operator, $r = |x|$ and $\tau = \frac{n+2}{n-2}$ is the so-called critical Sobolev exponent. The function $R(r)$ is the projection of the original R in equations (8.48) and (8.47); and this projection does not change the monotonicity of the function. The function R is also bounded and continuous, and we assume these throughout the subsection.

For a radial function R satisfying (8.53), there are three possibilities:
i) R is non-positive everywhere,
ii) $R \not\equiv C$ is nonnegative everywhere and monotone,
iii) $R > 0$ and non-increasing for $r < r_o$, $R \leq 0$ for $r > r_o$; or vice versa.

The first two cases violate the Kazdan-Warner type conditions, hence there is no solution. Thus, we only need to show that in the remaining case, there is also no solution. Without loss of generality, we may assume that:

$$R(r) > 0, R'(r) \leq 0 \text{ for } r < 1; \quad R(r) \leq 0 \text{ for } r \geq 1. \tag{8.57}$$

The proof of Theorem 8.4.1 is based on the following comparison formulas.

Lemma 8.4.1 *Let R be a continuous function satisfying (8.57). Let u be a solution of (8.54) and (8.56) for $n = 2$, then*

$$u(\lambda x) > u(\frac{\lambda x}{|x|^2}) - 4\ln|x| \quad \forall x \in B_1(0),\ \ 0 < \lambda \leq 1 \tag{8.58}$$

Lemma 8.4.2 *Let R be a continuous function satisfying (8.57). Let u be a solution of (8.55) and (8.56) for $n > 2$, then*

$$u(\lambda x) > |x|^{2-n}u(\frac{\lambda x}{|x|^2}) \quad \forall x \in B_1(0),\ \ 0 < \lambda \leq 1 \tag{8.59}$$

We first prove Lemma 8.4.2. A similar idea will work for Lemma 8.4.1.

Proof of Lemma 8.4.2. We use a new idea called the method of 'moving spheres '. Let $x \in B_1(0)$, then $\lambda x \in B_\lambda(0)$. The reflection point of λx about the sphere $\partial B_\lambda(0)$ is $\frac{\lambda x}{|x|^2}$. We compare the values of u at those pairs of points λx and $\frac{\lambda x}{|x|^2}$. In step 1, we start with the unit sphere and show that (8.59) is true for $\lambda = 1$. This is done by Kelvin transform and the strong maximum principle. Then in step 2, we move (shrink) the sphere $\partial B_\lambda(0)$

towards the origin. We show that one can always shrink $\partial B_\lambda(0)$ a little bit before reaching the origin. By doing so, we establish the inequality for all $x \in B_1(0)$ and $\lambda \in (0,1]$.

Step 1. In this step, we establish the inequality for $x \in B_1(0), \lambda = 1$. We show that

$$u(x) > |x|^{2-n} u(\frac{x}{|x|^2}) \;\; \forall x \in B_1(0). \tag{8.60}$$

Let $v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$ be the Kelvin transform. Then it is easy to verify that v satisfies the equation

$$-\triangle v = R(\frac{1}{r}) v^\tau \tag{8.61}$$

and v is regular.

It follows from (8.57) that $\triangle u < 0$, and $\triangle v \geq 0$ in $B_1(0)$. Thus $-\triangle(u - v) > 0$. Applying the maximum principle, we obtain

$$u > v \text{ in } B_1(0) \tag{8.62}$$

which is equivalent to (8.60).

Step 2. In this step, we move the sphere $\partial B_\lambda(0)$ towards $\lambda = 0$. We show that the moving can not be stopped until λ reaches the origin. This is done again by the Kelvin transform and the strong maximum principle.

Let $u_\lambda(x) = \lambda^{\frac{n}{2}-1} u(\lambda x)$. Then

$$-\triangle u_\lambda = R(\lambda x) u_\lambda^\tau(x). \tag{8.63}$$

Let $v_\lambda(x) = |x|^{2-n} u_\lambda(\frac{x}{|x|^2})$ be the Kelvin transform of u_λ. Then it is easy to verify that

$$-\triangle v_\lambda = R(\frac{\lambda}{r}) v_\lambda^\tau(x). \tag{8.64}$$

Let $w_\lambda = u_\lambda - v_\lambda$. Then by (8.63) and (8.64),

$$\triangle w_\lambda + R(\frac{\lambda}{r}) \psi_\lambda w_\lambda = [R(\frac{\lambda}{r}) - R(\lambda r)] u_\lambda^\tau \tag{8.65}$$

where ψ_λ is some function with values between $\tau u_\lambda^{\tau-1}$ and $\tau v_\lambda^{\tau-1}$. Taking into account of assumption (2.4), we have

$$R(\frac{\lambda}{r}) - R(\lambda r) \leq 0 \qquad \text{for } r \leq 1, \lambda \leq 1.$$

It follows from (8.65) that

$$\triangle w_\lambda + C_\lambda(x) w_\lambda \leq 0 \tag{8.66}$$

where $C_\lambda(x)$ is a bounded function if λ is bounded away from 0.

It is easy to see that for any λ, strict inequality holds somewhere for (8.66). Thus applying the strong maximum principle, we know that the inequality (8.59) is equivalent to

$$w_\lambda \geq 0. \tag{8.67}$$

From step 1, we know (8.67) is true for $(x, \lambda) \in B_1(0) \times \{1\}$. Now we decrease λ. Suppose (8.67) does not hold for all $\lambda \in (0, 1]$ and let $\lambda_o > 0$ be the smallest number such that the inequality is true for $(x, \lambda) \in B_1(0) \times [\lambda_o, 1]$. We will derive a contradiction by showing that for λ close to and less than λ_o, the inequality is still true. In fact, we can apply the strong maximum principle and then the Hopf lemma to (8.66) for $\lambda = \lambda_o$ to get:

$$w_{\lambda_o} > 0 \text{ in } B_1 \text{ and } \frac{\partial w_{\lambda_o}}{\partial r} < 0 \text{ on } \partial B_1. \tag{8.68}$$

These combined with the fact that $w_\lambda \equiv 0$ on ∂B_1 imply that (8.67) holds for λ close to and less than λ_o.

Proof of Lemma 8.4.1. In step 1, we let

$$v(x) = u(\frac{x}{|x|^2}) - 4 \ln |x|.$$

In step 2, let

$$u_\lambda(x) = u(\lambda x) + 2 \ln \lambda, \quad v_\lambda(x) = u_\lambda(\frac{x}{|x|^2}) - 4 \ln |x|.$$

Then arguing as in the proof of Lemma 8.4.2 we derive the conclusion of Lemma 8.4.1.

Now we are ready to prove Theorem 8.4.1.

Proof of Theorem 8.4.1. Taking λ to 0 in (8.58), we get $\ln |x| > 0$ for $|x| < 1$ which is impossible. Letting $\lambda \rightarrow 0$ in (8.59) and using the fact that $u(0) > 0$ we obtain again a contradiction. These complete the proof of Theorem 8.4.1.

Proof of Theorem 8.4.2. Noting that in the proof of Theorem 8.4.1, we only compare the values of the solutions along the same radial direction, one can easily adapt the argument in the proof of Theorem 8.4.1 to derive the conclusion of Theorem 8.4.2.

8.5 Method of Moving Planes in Integral Forms and Symmetry of Solutions for an Integral Equation

Let R^n be the $n-$dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{(n+\alpha)/(n-\alpha)} dy. \tag{8.69}$$

It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities. In [L], Lieb classified the maximizers of the functional, and thus obtained the best constant in the H-L-S inequalities. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (8.69) as an open problem.

This integral equation is also closely related to the following family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2}u = u^{(n+\alpha)/(n-\alpha)}. \tag{8.70}$$

In the special case $n \geq 3$ and $\alpha = 2$, it becomes

$$-\Delta u = u^{(n+2)/(n-2)}. \tag{8.71}$$

As we mentioned in Section 6, solutions to this equation were studied by Gidas, Ni, and Nirenberg [GNN]; Caffarelli, Gidas, and Spruck [CGS]; Chen and Li [CL]; and Li [Li]. Recently, Wei and Xu [WX] generalized this result to the solutions of (8.70) with α being any even numbers between 0 and n.

Apparently, for other real values of α between 0 and n, equation (8.70) is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx / (\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx)^{\frac{n-\alpha}{n}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{\frac{\alpha}{2}}(R^n)$ to $L^{\frac{2n}{n-\alpha}}(R^n)$:

$$(\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx)^{\frac{n-\alpha}{n}} \leq C \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx.$$

As usual, here $(-\Delta)^{\alpha/2}$ is defined by Fourier transform. We can show that (see [CLO]), all the solutions of partial differential equation (8.70) satisfy our integral equation (8.69), and vise versa. Therefore, to classify the solutions of this family of partial differential equations, we only need to work on the integral equations (8.69).

In order either the Hardy-Littlewood-Sobolev inequality or the above mentioned critical Sobolev imbedding to make sense, u must be in $L^{\frac{2n}{n-\alpha}}(R^n)$. Under this assumption, we can show that (see [CLO]) a positive solution u is in fact bounded, and therefore possesses higher regularity. Furthermore, by using the method of moving planes, we can show that if a solution is locally $L^{\frac{2n}{n-\alpha}}$, then it is in $L^{\frac{2n}{n-\alpha}}(R^n)$. Hence, in the following, we call a solution u regular if it is locally $L^{\frac{2n}{n-\alpha}}$. It is interesting to notice that if this condition is violated, then a solution may not be bounded. A simple example is $u = \frac{1}{|x|^{(n-\alpha)/2}}$, which is a singular solution. We studied such solutions in [CLO1].

We will use the method of moving planes to prove

Theorem 8.5.1 *Every positive regular solution $u(x)$ of (8.69) is radially symmetric and decreasing about some point x^o and therefore assumes the form*

$$c(\frac{t}{t^2+|x-x^o|^2})^{(n-\alpha)/2}, \tag{8.72}$$

with some positive constants c and t.

To better illustrate the idea, we will first prove the Theorem under stronger assumption that $u \in L^{\frac{2n}{n-2}}(R^n)$. Then we will show how the idea of this proof can be extended under weaker condition that u is only locally in $L^{\frac{2n}{n-2}}$.

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, \dots, x_n) \mid x_1 \leq \lambda\},$$

and let $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ and $u_\lambda(x) = u(x^\lambda)$. (Again see the previous Figure 1.)

Lemma 8.5.1 *For any solution $u(x)$ of (1.1), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} (\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}})(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}})dy, \tag{8.73}$$

It is also true for $v(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2})$, the Kelvin type transform of $u(x)$, for any $x \neq 0$.

Proof. Since $|x - y^\lambda| = |x^\lambda - y|$, we have

$$u(x) = \int_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-\alpha}} u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}} dy, \text{ and}$$
$$u(x^\lambda) = \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\alpha}} u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}.$$

This implies (8.73). The conclusion for $v(x)$ follows similarly since it satisfies the same integral equation (8.69) for $x \neq 0$.

Proof of the Theorem 8.5.1 (*Under Stronger Assumption* $u \in L^{\frac{2n}{n-2}}(R^n)$).

Let $w_\lambda(x) = u_\lambda(x) - u(x)$. As in Section 6, we will first show that, for λ sufficiently negative,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda. \tag{8.74}$$

Then we can start moving the plane $T_\lambda = \partial \Sigma_\lambda$ from near $-\infty$ to the right, as long as inequality (8.74) holds. Let

$$\lambda_o = \sup\{\lambda \leq 0 \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\}.$$

We will show that

$$\lambda_o < \infty, \quad \text{and } w_{\lambda_o}(x) \equiv 0.$$

Unlike partial differential equations, here we do not have any differential equations for w_λ. Instead, we will introduce a new idea – the method of moving planes in integral forms. We will exploit some global properties of the integral equations.

Step 1. Start Moving the Plane from near $-\infty$.

Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}. \tag{8.75}$$

We show that for sufficiently negative values of λ, Σ_λ^- must be empty. By Lemma 8.5.1, it is easy to verify that

$$\begin{aligned}
u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) [u^\tau(y) - u_\lambda^\tau(y)]dy \\
&\leq \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} [u^\tau(y) - u_\lambda^\tau(y)]dy \\
&= \tau \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \psi_\lambda^{\tau-1}(y)[u(y) - u_\lambda(y)]dy \\
&\leq \tau \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} u^{\tau-1}(y)[u(y) - u_\lambda(y)]dy,
\end{aligned}$$

where $\psi_\lambda(x)$ is valued between $u_\lambda(x)$ and $u(x)$ by the *Mean Value Theorem*, and since on Σ_λ^-, $u_\lambda(x) < u(x)$, we have $\psi_\lambda(x) \leq u(x)$.

We first apply the classical Hardy-Littlewood-Sobolev inequality (its equivalent form) to the above to obtain, for any $q > \frac{n}{n-\alpha}$:

$$\begin{aligned}
\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\leq C \|u^{\tau-1} w_\lambda\|_{L^{nq/(n+\alpha q)}(\Sigma_\lambda^-)} \\
&= \left(\int_{\Sigma_{\lambda^-}} |u^{\tau-1}(x)|^{\frac{nq}{n+\alpha q}} |w_\lambda(x)|^{\frac{nq}{n+\alpha q}} dx \right)^{\frac{n+\alpha q}{nq}}.
\end{aligned}$$

Then use the Hölder inequality to the last integral in the above. First choose

$$s = \frac{n + \alpha q}{n}$$

so that

$$\frac{nq}{n+\alpha q} \cdot s = q,$$

then choose

$$r = \frac{n + \alpha q}{\alpha q}$$

to ensure

$$\frac{1}{r} + \frac{1}{s} = 1.$$

We thus arrive at

$$\begin{aligned}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\leq C\{\int_{\Sigma_\lambda^-} u^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \\ &\leq C\{\int_{\Sigma_\lambda} u^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}\end{aligned}$$

By condition that $u \in L^{\tau+1}(R^n)$, we can choose N sufficiently large, such that for $\lambda \leq -N$, we have

$$C\{\int_{\Sigma_\lambda} u^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now (8.76) implies that $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be measure zero, and hence empty by the continuity of $w_\lambda(x)$. This verifies (8.74) and hence completes *Step 1.* The readers can see that here we have actually used Corollary 7.5.1, with $\Omega = \Sigma_\lambda$, a region *near infinity.*

Step 2. We now move the plane $T_\lambda = \{x \mid x_1 = \lambda\}$ to the right as long as (8.74) holds. Let λ_o as defined before, then we must have $\lambda_o < \infty$. This can be seen by applying a similar argument as in *Step 1* from λ near $+\infty$.

Now we show that

$$w_{\lambda_o}(x) \equiv 0\ , \quad \forall x \in \Sigma_{\lambda_o}. \tag{8.76}$$

Otherwise, we have $w_{\lambda_o}(x) \geq 0$, but $w_{\lambda_o}(x) \not\equiv 0$ on Σ_{λ_o}; we show that the plane can be moved further to the right. More precisely, there exists an ϵ depending on n, α, and the solution $u(x)$ itself such that $w_\lambda(x) \geq 0$ on Σ_λ for all λ in $[\lambda_o, \lambda_o + \epsilon)$.

By Lemma 8.5.1, we have in fact $w_{\lambda_o}(x) > 0$ in the interior of Σ_{λ_o}. Let $\overline{\Sigma_{\lambda_o}^-} = \{x \in \Sigma_{\lambda_o} | \, w_{\lambda_o}(x) \leq 0\}$. Then obviously, $\overline{\Sigma_{\lambda_o}^-}$ has measure zero, and $\lim_{\lambda \to \lambda_o} \Sigma_\lambda^- \subset \overline{\Sigma_{\lambda_o}^-}$. From the first inequality of (8.76), we deduce,

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C\{\int_{\Sigma_\lambda^-} u^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \tag{8.77}$$

Condition $u \in L^{\tau+1}(R^n)$ ensures that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$C\{\int_{\Sigma_\lambda^-} u^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now by (8.77), we have $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be empty. Here we have actually applied Corollary 7.5.1 with $\Omega = \Sigma_\lambda^-$. For λ sufficiently close to λ_o, Ω is contained in a very *narrow region.*

Proof of the Theorem 8.5.1 (*under Weaker Assumption that u is only locally in* $L^{\tau+1}$).

Outline. Since we do not assume any integrability condition of $u(x)$ near infinity, we are not able to carry on the method of moving planes directly on $u(x)$. To overcome this difficulty, we consider $v(x)$, the Kelvin type transform of $u(x)$. It is easy to verify that $v(x)$ satisfies the same equation (8.69), but has a possible singularity at origin, where we need to pay some particular attention. Since u is locally $L^{\tau+1}$, it is easy to see that $v(x)$ has no singularity at infinity, i.e. for any domain Ω that is a positive distance away from the origin,

$$\int_{\Omega} v^{\tau+1}(y)\,dy < \infty. \tag{8.78}$$

Let λ be a real number and let the moving plane be $x_1 = \lambda$. We compare $v(x)$ and $v_\lambda(x)$ on $\Sigma_\lambda \setminus \{0\}$. The proof consists of three steps. In step 1, we show that there exists an $N > 0$ such that for $\lambda \leq -N$, we have

$$v(x) \leq v_\lambda(x), \quad \forall x \in \Sigma_\lambda \setminus \{0\}. \tag{8.79}$$

Thus we can start moving the plane continuously from $\lambda \leq -N$ to the right as long as (8.79) holds. If the plane stops at $x_1 = \lambda_o$ for some $\lambda_o < 0$, then $v(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_o$. This implies that $v(x)$ has no singularity at the origin and $u(x)$ has no singularity at infinity. In this case, we can carry on the moving planes on $u(x)$ directly to obtain the radial symmetry and monotonicity. Otherwise, we can move the plane all the way to $x_1 = 0$, which is shown in step 2. Since the direction of x_1 can be chosen arbitrarily, we deduce that $v(x)$ must be radially symmetric and decreasing about the origin. We will show in step 3 that, in any case, $u(x)$ can not have a singularity at infinity, and hence both u and v are in $L^{\tau+1}(R^n)$.

Step 1 and *Step 2* are entirely similar to the proof under stronger assumption. What we need to do is to replace u there by v and Σ_λ there by $\Sigma_\lambda \setminus \{0\}$. Hence we only show *Step 3* here.

Step 3. Finally, we showed that u has the desired asymptotic behavior at infinity, i.e., it satisfies

$$u(x) = O(\frac{1}{|x|^{n-2}}).$$

Suppose in the contrary, let x^1 and x^2 be any two points in R^n and let x^o be the midpoint of the line segment $\overline{x^1 x^2}$. Consider the Kelvin type transform centered at x^o:

$$v(x) = \frac{1}{|x - x^o|^{n-\alpha}} u(\frac{x - x^o}{|x - x^o|^2}).$$

Then $v(x)$ has a singularity at x^o. Carry on the arguments as in Steps 1 and 2, we conclude that $v(x)$ must be radially symmetric about x^o, and in particular, $u(x^1) = u(x^2)$. Since x^1 and x^2 are any two points in R^n, u must be constant. This is impossible. Similarly, The continuity and higher regularity of u follows from standard theory on singular integral operators. This completes the proof of the Theorem.

A

Appendices

A.1 Notations

A.1.1 Algebraic and Geometric Notations

1. $A = (a_{ij})$ is a an $m \times n$ matrix with ij^{th} entry a_{ij}.
2. $det(a_{ij})$ is the determinant of the matrix (a_{ij}).
3. A^T = transpose of the matrix A.
4. R^n = n-dimensional real Euclidean space with a typical point $x = (x_1, \cdots, x_n)$.
5. Ω usually denotes and open set in R^n .
6. $\bar{\Omega}$ is the closure of Ω.
7. $\partial\Omega$ is the boundary of Ω.
8. For $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in R^n,
$$x \cdot y = \sum_{i=1}^{n} x_i y_i, \quad |x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$
9. $|x - y|$ is the distance of the two points x and y in R^n.
10. $B_r(x^o) = \{x \in R^n \mid |x - x^o| < r\}$ is the ball of radius r centered at the point x^o in R^n.
11. S^n is the sphere of radius one centered at the origin in R^{n+1}, i.e. the boundary of the ball of radius one centered at the origin:
$$B_1(0) = \{x \in R^{n+1} \mid |x| < 1\}.$$

A.1.2 Notations for Functions and Derivatives

1. For a function $u : \Omega \subset R^n \to R^1$, we write
$$u(x) = u(x_1, \cdots, x_n) \;\; , \;\; x \in \Omega.$$
We say u is smooth if it is infinitely differentiable.

2. For two functions u and v, $u \equiv v$ means u is identically equal to v.
3. We set
$$u := v$$
to define u as equaling v.
4. We usually write u_{x_i} for $\frac{\partial u}{\partial x_i}$, the first partial derivative of u with respect to x_i. Similarly
$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \quad u_{x_i x_j x_k} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} \quad .$$
5.
$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$
where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of order
$$|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n.$$
6. For a non-negative integer k,
$$D^k u(x) := \{D^\alpha u(x) \mid |\alpha| = k\}$$
the set of all partial derivatives of order k.
If $k = 1$, we regard the elements of Du as being arranged in a vector
$$Du := (u_{x_1}, \cdots, u_{x_n}) = \bigtriangledown u,$$
the gradient vector.
If $k = 2$, we regard the elements of $D^2 u$ being arranged in a matrix
$$D^2 u := \begin{pmatrix} \frac{\partial^2 u}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \cdot & \cdot & \cdot \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n \partial x_n} \end{pmatrix},$$
the Hessian matrix.
7. The Laplacian
$$\triangle u := \sum_{i=1}^{n} u_{x_i x_i}$$
is the trace of $D^2 u$.
8.
$$|D^k u| := \left(\sum_{|\alpha| = k} |D^\alpha u|^2 \right)^{1/2} .$$

A.1.3 Function Spaces

1. $$C(\Omega) := \{u : \Omega \to R^1 \mid u \text{ is continuous } \}$$

2. $$C(\bar{\Omega}) := \{u \in C(\Omega) \mid u \text{ is uniformly continuous}\}$$

3. $$\|u\|_{C(\bar{\Omega})} := \sup_{x \in \Omega} |u(x)|$$

4. $$C^k(\Omega) := \{u : \Omega \to R^1 \mid u \text{ is } k\text{-times continuously differentiable } \}$$

5. $$C^k(\bar{\Omega}) := \{u \in C^k(\Omega) \mid D^\alpha u \text{ is uniformly continuous for all } |\alpha| \leq k\}$$

6. $$\|u\|_{C^k(\bar{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{\Omega})}$$

7. Assume Ω is an open subset in R^n and $0 < \alpha \leq 1$. If there exists a constant C such that
$$|u(x) - u(y)| \leq C|x - y|^\alpha \;\; , \;\; \forall \, x \, , y \in \Omega,$$
then we say that u is *Höder* continuous in Ω.
8. The α^{th} *Höder* semi-norm of u is
$$[u]_{C^{0,\alpha}(\bar{\Omega})} := \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}$$
9. The α^{th} *Höder* norm is
$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} := \|u\|_{C(\bar{\Omega})} + [u]_{C^{0,\alpha}(\bar{\Omega})}$$
10. The *Höder* space $C^{k,\alpha}(\bar{\Omega})$ consists of all functions $u \in C^k(\bar{\Omega})$ for which the norm
$$\|u\|_{C^{k,\alpha}(\bar{\Omega})} := \sum_{|\beta| \leq k} \|D^\beta u\|_{C(\bar{\Omega})} + \sum_{|\beta| = k} [D^\beta u]_{C^{0,\alpha}(\bar{\Omega})}$$
is finite.
11. $C^\infty(\Omega)$ is the set of all infinitely differentiable functions
12. $C_0^k(\Omega)$ denotes the subset of functions in $C^k(\Omega)$ with compact support in Ω.

13. $L^p(\Omega)$ is the set of Lebesgue measurable functions u on Ω with

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p} < \infty.$$

14. $L^\infty(\Omega)$ is the set of Lebesgue measurable functions u with

$$\|u\|_{L^\infty(\Omega)} := ess \sup_\Omega |u| < \infty.$$

15. $L^p_{loc}(\Omega)$ denotes the set of Lebesgue measurable functions u on Ω whose p^{th} power is locally integrable, i.e. for any compact subset $K \subset \Omega$,

$$\int_K |u(x)|^p dx < \infty.$$

16. $W^{k,p}(\Omega)$ denotes the Sobolev space consisting functions u with

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{0\leq|\alpha|\leq k} \left(\int_\Omega |D^\alpha u|^p dx\right)^{1/p} < \infty.$$

17. $H^{k,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ under the norm $\|u\|_{W^{k,p}(\Omega)}$.
18. For any open subset $\Omega \subset R^n$, any non-negative integer k and any $p \geq 1$, it is shown (See Adams [Ad]) that

$$H^{k,p}(\Omega) = W^{k,p}(\Omega).$$

19. For $p = 2$, we usually write

$$H^k(\Omega) = H^{k,2}(\Omega).$$

A.1.4 Notations for Estimates

1. In the process of estimates, we use C to denote various constants that can be explicitly computed in terms of known quantities. The value of C may change from line to line in a given computation.
2. We write

$$u = O(v) \quad \text{as } x{\rightarrow}x^o,$$

provided there exists a constant C such that

$$|u(x)| \leq C|v(x)|$$

for x sufficiently close to x^o.
3. We write

$$u = o(v) \quad \text{as } x{\rightarrow}x^o,$$

provided

$$\frac{|u(x)|}{|v(x)|}{\rightarrow}0 \quad \text{as } x{\rightarrow}x^o.$$

A.1.5 Notations from Riemannian Geometry

1. M denotes a differentiable manifold.
2. T_pM is the tangent space of M at point p.
3. T_p^*M is the co-tangent space of M at point p.
4. A Riemannian metric g can be expressed in a matrix

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdot & \cdot & \cdot \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

where in a local coordinates

$$g_{ij}(p) = < \left(\frac{\partial}{\partial x_i}\right)_p , \left(\frac{\partial}{\partial x_j}\right)_p >_p .$$

5. (M, g) denotes an n-dimensional manifold M with Riemannain metric g. In a local coordinates, the length element

$$ds := \left(\sum_{i,j=1}^{n} g_{ij} dx_i dx_j\right)^{1/2} .$$

The volume element

$$dV := \sqrt{\det(g_{ij})} dx_1 \cdots dx_n .$$

6. $K(x)$ denotes the Gaussian curvature, and $R(x)$ scalar curvature of a manifold M at point x.
7. For two differentiable vector fields X and Y on M, the Lie Bracket is

$$[X, Y] := XY - YX.$$

8. D denotes an affine connection on a differentiable manifold M.
9. A vector field $V(t)$ along a curve $c(t) \in M$ is parallel if $\frac{DV}{dt} = 0$.
10. Given a Riemannian manifold M with metric $<,>$, there exists a unique connection D on M that is compatible with the metric $<,>$, i.e.

$$< X, Y >_{c(t)} = \text{ constant } ,$$

for any pair of parallel vector fields X and Y along any smooth curve $c(t)$.
11. In a local coordinates $(x_1, \cdots, x_n)$, this unique Riemannian connection can be expressed as

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

where Γ_{ij}^k is the Christoffel symbols.

12. We denote
$$\bigtriangledown_i = D_{\frac{\partial}{\partial x_i}}.$$
13. The summation convention. We write, for instance,
$$X^i \frac{\partial}{\partial x_i} := \sum_i X^i \frac{\partial}{\partial x_i},$$
$$\Gamma^j_{ik} dx^k := \sum_k \Gamma^j_{ik} dx^k,$$
and so on.
14. A smooth function $f(x)$ on M is a $(0,0)$-tensor. $\bigtriangledown f$ is a $(1,0)$-tensor defined by
$$\bigtriangledown f = \bigtriangledown_i f dx^i = \frac{\partial f}{\partial x_i} dx^i.$$
$\bigtriangledown^2 f$ is a $(2,0)$-tensor:
$$\begin{aligned}\bigtriangledown^2 f &= \bigtriangledown_j \left(\frac{\partial f}{\partial x_i} dx^i \right) \otimes dx^j \\ &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \right) dx^i \otimes dx^j.\end{aligned}$$
In the Riemannian context, $\bigtriangledown^2 f$ is called the Hessian of f and denoted by Hess(f). Its ij^{th} component is
$$(\bigtriangledown^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k}.$$
15. The trace of the Hessian matrix $((\bigtriangledown^2 f)_{ij})$ is defined to be the Laplace-Beltrami operator $\triangle$.
$$\triangle = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial x_j})$$
where $|g|$ is the determinant of (g_{ij}).
16. We abbreviate:
$$\int_M \bigtriangledown u \bigtriangledown v dV_g := \int_M < \bigtriangledown u, \bigtriangledown v > dV_g$$
where
$$< \bigtriangledown u, \bigtriangledown v >= g^{ij} \bigtriangledown_i u \bigtriangledown_j v = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}$$
is the scalar product associated with g for 1-forms.

17. The norm of the k^{th} covariant derivative of u, $|\bigtriangledown^k u|$ is defined in a local coordinates chart by

$$|\bigtriangledown^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\bigtriangledown^k u)_{i_1 \cdots i_k} (\bigtriangledown^k u)_{j_1 \cdots j_k}.$$

In particular, we have

$$|\bigtriangledown^1 u|^2 = |\bigtriangledown u|^2 = g^{ij} (\bigtriangledown u)_i (\bigtriangledown u)_j = g^{ij} \bigtriangledown_i u \bigtriangledown_j u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

18. $\mathcal{C}_k^p(M)$ is the space of C^∞ functions on M such that

$$\int_M |\bigtriangledown^j u|^p dV_g < \infty \,, \forall\, j = 0, 1, \cdots, k.$$

19. The Sobolev space $H^{k,p}(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect to the norm

$$\|u\|_{H^{k,p}(M)} = \sum_{j=0}^{k} \left(\int_M |\bigtriangledown^j u|^p dV_g \right)^{1/p}.$$

A.2 Inequalities

Young's Inequality. Assume $1 < p, q < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $a, b > 0$, it holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{A.1}$$

Proof. We will use the convexity of the exponential function e^x. Let $x_1 < x_2$ be any two point in R^1. Then since $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} x_1 + \frac{1}{q} x_2$ is a point between x_1 and x_2. By the convexity of e^x, we have

$$e^{\frac{1}{p} x_1 + \frac{1}{q} x_2} \leq \frac{1}{p} e^{x_1} + \frac{1}{q} e^{x_2}. \tag{A.2}$$

Taking $x_1 = \ln a^p$ and $x_2 = \ln b^q$ in (A.2), we arrive at inequality (A.1). □

Cauchy-Schwarz Inequality.

$$|x \cdot y| \leq |x||y| \,, \;\; \forall\, x, y \in R^n.$$

Proof. For any real number $\lambda > 0$, we have

$$0 \leq |x \pm \lambda y|^2 = |x|^2 \pm 2\lambda x \cdot y + \lambda^2 |y|^2.$$

It follows that

$$\pm x \cdot y \leq \frac{1}{2\lambda} |x|^2 + \frac{\lambda}{2} |y|^2.$$

The minimum value of the right hand side is attained at $\lambda = \frac{|x|}{|y|}$. At this value of λ, we arrive at desired inequality. □

Höder's **Inequality.** Let Ω be a domain in R^n. Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_\Omega |u\,v| dx \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{A.3}$$

Proof. Let $f = \frac{u}{\|u\|_{L^p(\Omega)}}$ and $g = \frac{v}{\|v\|_{L^q(\Omega)}}$. Then obviously, $\|f\|_{L^p(\Omega)} = 1 = \|g\|_{L^q(\Omega)}$. It follows from the Young's inequality (A.1),

$$\int_\Omega |f\,g| dx \le \frac{1}{p} \int_\Omega |f|^p dx + \frac{1}{q} \int_\Omega |g|^q dx = \frac{1}{p} + \frac{1}{q} = 1.$$

This implies (A.3). □

Minkowski's Inequality. Assume $1 \le p \le \infty$. Then for any $u, v \in L^p(\Omega)$,

$$\|u + v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Proof. By the above *Höder* inequality, we have

$$\begin{aligned}
\|u+v\|^p_{L^p(\Omega)} &= \int_\Omega |u+v|^p dx \le \int_\Omega |u+v|^{p-1}(|u| + |v|) dx \\
&\le \left(\int_\Omega |u+v|^p dx \right)^{\frac{p-1}{p}} \left[\left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}} + \left(\int_\Omega |v|^p dx \right)^{\frac{1}{p}} \right] \\
&= \|u+v\|^{p-1}_{L^p(\Omega)} \left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right).
\end{aligned}$$

Then divide both sides by $\|u+v\|^{p-1}_{L^p(\Omega)}$. □

Interpolation Inequality for L^p Norms. Assume that $u \in L^r(\Omega) \cap L^s(\Omega)$ with $1 \le r \le s \le \infty$. Then for any $r \le t \le s$ and $0 < \theta < 1$ such that

$$\frac{1}{t} = \frac{\theta}{r} + \frac{1-\theta}{s} \tag{A.4}$$

we have $u \in L^t(\Omega)$, and

$$\|u\|_{L^t(\Omega)} \le \|u\|^\theta_{L^r(\Omega)} \|u\|^{1-\theta}_{L^s(\Omega)}. \tag{A.5}$$

Proof. Write

$$\int_\Omega |u|^t dx = \int_\Omega |u|^a \cdot |u|^{t-a} dx.$$

Choose p and q so that

$$ap = r\,, (t-a)q = s\,, \text{ and } \frac{1}{p} + \frac{1}{q} = 1. \tag{A.6}$$

Then by *Höder* inequality, we have

$$\int_{\Omega} |u|^t dx \leq \left(\int_{\Omega} |u|^r dx \right)^{a/r} \cdot \left(\int_{\Omega} |u|^s dx \right)^{(t-a)/s}.$$

Divide through the powers by t, we obtain

$$\|u\|_t \leq \|u\|_r^{a/t} \cdot \|u\|_s^{1-a/t}.$$

Now let $\theta = a/t$, we arrive at desired inequality (A.5). Here the restriction (A.4) comes from (A.6). □

A.3 Calderon-Zygmund Decomposition

Lemma A.3.1 *For $f \in L^1(R^n)$, $f \geq 0$, fixed $\alpha > 0$, $\exists \quad E,\ G$ such that*
(i) $R^n = E \cup G,\ E \cap G = \emptyset$
(ii) $f(x) \leq \alpha$, a.e. $x \in E$
(iii) $G = \bigcup_{k=1}^{\infty} Q_k$, $\{Q_k\}$: disjoint cubes s.t.

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f(x)dx \leq 2^n \alpha$$

Proof. Since $\int_{R^n} f(x)dx$ is finite, for a given $\alpha > 0$, one can pick a cube Q_o sufficiently large, such that

$$\int_{Q_o} f(x)dx \leq \alpha |Q_o|. \tag{A.7}$$

Divide Q_o into 2^n equal sub-cubes with disjoint interior. Those sub-cubes Q satisfying

$$\int_Q f(x)dx \leq \alpha |Q|$$

are similarly sub-divided, and this process is repeated indefinitely. Let $\mathcal{Q}$ denote the set of sub-cubes Q thus obtained that satisfy

$$\int_Q f(x)dx > \alpha |Q|.$$

For each $Q \in \mathcal{Q}$, let $\tilde{Q}$ be its predecessor, i.e., Q is one of the 2^n sub-cubes of $\tilde{Q}$. Then obviously, we have $|\tilde{Q}|/|Q| = 2^n$, and consequently,

$$\alpha < \frac{1}{|Q|} \int_Q f(x)dx \leq \frac{1}{|Q|} \int_{\tilde{Q}} f(x)dx \leq \frac{1}{|Q|} \alpha |\tilde{Q}| = 2^n \alpha. \tag{A.8}$$

Let

$$G = \bigcup_{Q \in \mathcal{Q}} Q \ , \quad \text{and } E = Q_o \setminus G.$$

Then iii) follows immediately. To see ii), noticing that each point of E lies in a nested sequence of cubes Q with diameters tending to zero and satisfying

$$\int_Q f(x)dx \leq \alpha |Q|,$$

now by Lebesgue's Differentiation Theorem, we have

$$f(x) \leq \alpha \ , \ \ a.e \text{ in } E. \tag{A.9}$$

This completes the proof of the Lemma. □

Lemma A.3.2 *Let T be a linear operator from $L^p(\Omega) \cap L^q(\Omega)$ into itself with $1 \leq p < q < \infty$. If T is of weak type (p, p) and weak type (q, q), then for any $p < r < q$, T is of strong type (r, r). More precisely, if there exist constants B_p and B_q, such that, for any $t > 0$,*

$$\mu_{Tf}(t) \leq \left(\frac{B_p \|f\|_p}{t} \right)^p \ \ \text{and} \ \ \mu_{Tf}(t) \leq \left(\frac{B_q \|f\|_q}{t} \right)^q \ , \ \ \forall f \in L^p(\Omega) \cap L^q(\Omega),$$

then

$$\|Tf\|_r \leq C B_p^\theta B_q^{1-\theta} \|f\|_r \ , \ \ \forall f \in L^p(\Omega) \cap L^q(\Omega),$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and C depends only on p, q, and r.

Proof. For any number $s > 0$, let

$$g(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq s \\ 0 & \text{if } |f(x)| > s. \end{cases}$$

We split f into the good part g and the bad part b: $f(x) = g(x) + b(x)$. Then

$$|Tf(x)| \leq |Tg(x)| + |Tb(x)|,$$

and hence

$$\begin{aligned} \mu(t) \equiv \mu_{Tf}(t) &\leq \mu_{Tg}(\frac{t}{2}) + \mu_{Tb}(\frac{t}{2}) \\ &\leq \left(\frac{2B_q}{t} \right)^q \int_\Omega |g(x)|^q dx + \left(\frac{2B_p}{t} \right)^p \int_\Omega |b(x)|^p dx. \end{aligned}$$

It follows that

$$\begin{aligned}
\int_\Omega |Tf|^r dx &= \int_0^\infty \mu(t) d(t^r) = r\int_0^\infty t^{r-1}\mu(t)dt \\
&\le r(2B_q)^q \int_0^\infty t^{r-1-q}\left(\int_{|f|\le s} |f(x)|^q dx\right) dt \\
&+ r(2B_p)^p \int_0^\infty t^{r-1-p}\left(\int_{|f|> s} |f(x)|^p dx\right) dt \\
&\equiv r(2B_q)^q I_q + r(2B_p)^p I_p. \qquad \text{(A.10)}
\end{aligned}$$

Let $s = t/A$ for some positive number A to be fixed later. Then

$$\begin{aligned}
I_q &= A^{r-q}\int_0^\infty s^{r-1-q}\left(\int_{|f|\le s}|f(x)|^q dx\right) ds \\
&= A^{r-q}\int_\Omega |f(x)|^q \left(\int_{|f|}^\infty s^{r-1-q} ds\right) dx \\
&= \frac{A^{r-q}}{q-r}\int_\Omega |f(x)|^r dx. \qquad \text{(A.11)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_p &= A^{r-p}\int_0^\infty s^{r-1-p}\left(\int_{|f|>s}|f(x)|^p dx\right) ds = \int_\Omega |f(x)|^p \int_0^{|f|} s^{r-1-p} ds \\
&= \frac{A^{r-p}}{r-p}\int_\Omega |f(x)|^r dx. \qquad \text{(A.12)}
\end{aligned}$$

Combining (A.10), (A.11), and (A.12), we derive

$$\int_\Omega |Tf(x)|^r dx \le rF(A)\int_\Omega |f(x)|^r dx, \qquad \text{(A.13)}$$

where

$$F(A) = \frac{(2B_q)^q A^{r-q}}{q-r} + \frac{(2B_p)^p A^{r-p}}{r-p}.$$

By elementary calculus, one can easily verify that the minimum of $F(A)$ is

$$A = 2B_q^{q/(q-p)} B_p^{p/(p-q)}.$$

For this value of A, (A.13) becomes

$$\int_\Omega |Tf(x)|^r dx \le r2^r \left(\frac{1}{q-r} + \frac{1}{r-p}\right) B_q^{q(r-p)/(q-p)} B_p^{p(q-r)/(q-p)} \int_\Omega |f(x)|^r dx.$$

Letting

$$C = 2\left(\frac{r}{q-r} + \frac{r}{r-p}\right)^{1/r}, \text{ and } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

we arrive immediately at

$$\|Tf\|_{L^r(\Omega)} \leq C B_p^\theta B_q^{1-\theta} \|f\|_{L^r(\Omega)}.$$

This completes the proof of the Lemma. □

A.4 The Contraction Mapping Principle

Let X be a linear space with norm $\|\cdot\|$, and let T be a mapping from X into itself. If there exists a number $\theta < 1$, such that

$$\|Tx - Ty\| \leq \theta \|x - y\| \quad \text{for all } x, y \in X,$$

Then T is called a *contraction mapping* .

Theorem A.4.1 *Let T be a contraction mapping in a Banach space $\mathcal{B}$. Then it has a unique fixed point in $\mathcal{B}$, that is, there is a unique $x \in \mathcal{B}$, such that*

$$Tx = x\,.$$

Proof. Let x_o be any point in the Banach space $\mathcal{B}$. Define

$$x_1 = Tx_o\,, \text{ and } x_{k+1} = Tx_k \quad k = 1, 2, \cdots.$$

We show that $\{x_k\}$ is a Cauchy sequence in $\mathcal{B}$. In fact, for any integers $n > m$,

$$\begin{aligned}
\|x_n - x_m\| &\leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \\
&= \sum_{k=m}^{n-1} \|T^k x_1 - T^k x_o\| \\
&\leq \sum_{k=m}^{n-1} \theta^k \|x_1 - x_o\| \\
&\leq \frac{\theta^m}{1-\theta} \|x_1 - x_o\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Since a Banach is complete, the sequence $\{x_k\}$ converges to an element x in $\mathcal{B}$. Taking limit on both side of

$$x_{k+1} = Tx_k\,,$$

we arrive at

$$Tx = x. \tag{A.14}$$

To see the uniqueness, assume that y is a solution of (A.14). Then

$$\|x-y\| = \|Tx - Ty\| \leq \theta \|x-y\|.$$

Therefore, we must have

$$\|x-y\| = 0,$$

That is $x = y$. □

A.5 The Arzela-Ascoli Theorem

We say that the sequence of functions $\{u_k(x)\}$ are uniformly equicontinuous if for each $\epsilon > 0$, there exists $\delta > 0$, such that

$$|u_k(x) - u_k(y)| < \epsilon \quad \text{whenever } |x-y| < \delta$$

for all $x, y \in R^n$ and $k = 1, 2, \cdots$.

Theorem A.5.1 *(Arzela-Ascoli Compactness Criterion for Uniform Convergence).*

Assume that $\{u_k(x)\}$ is a sequence of real-valued functions defined on R^n satisfying

$$|u_k(x)| \leq M \quad k = 1, 2, \cdots, \quad x \in R^n$$

for some positive number M, and $\{u_k(x)\}$ are uniformly equicontinuous. Then there exists a subsequence $\{u_{k_i}(x)\}$ of $\{u_k(x)\}$ and a continuous function $u(x)$, such that

$$u_{k_i} \to u$$

uniformly on any compact subset of R^n.

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